VALUE AND FAIRNESS

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1. INTRODUCTION

Value allocation is a cooperative solution concept designed to predict which Pareto optimal outcomes will be selected by an explicit bargaining process among the agents in an economy. The concept of value allocation may be either in the context of cardinal utility due to Shapley (1969) or in an ordinal formulation due to Aumann (1975). The controversial features of value allocations have been recently examined by a number of authors, notably, Shafer (1980), Roth (1980, 1983), Harsanyi (1980), Yannelis (1983), Thomson (1983), Aumann (1983) and Scafuri-Yannelis (1984). This research has also pointed out several peculiarities of value allocations.

It is the purpose of the present notes to further analyze several properties of the above concept by means of examples. Also, we will compare the concept of value allocation with three other solution concepts, i.e., competitive equilibrium, core, and nondiscrimination. Although in an economy with an atomless measure space of economic agents all the above solution concepts coincide, however, in an economy with finitely many agents none of these concepts need be the same.

The paper proceeds as follows: Section 2 discusses the relationship of value, core, competitive equilibrium and nondiscrimination in a finite exchange economy framework. Sections 3, 4 and 5 investigate several properties of value allocations, i.e., their manipulability, fairness and symmetry. In Section 6 the concept of nondiscrimination is analyzed in a mixed finite exchange economy setting. Finally, Section 7 shows how results for finite economies can be extended to economies with an atomless measure space of economic agents.

2. PRELIMINARY RESULTS

2.1. *Notation*

R denotes the set of real numbers

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R^ℓ denotes the ℓ-fold Cartesian product of R

R_+^ℓ denotes the positive cone of R^ℓ

For any x, y in R^ℓ,

$x \geq y$ means $x_i \geq y_i$ for all i

$x > y$ means $x \geq y$ and $x \neq y$

$x >> y$ means $x_i > y_i$ for all i

$$x \cdot y = \sum_{i=1}^{\ell} x_i y_i$$

$$\|x\| = \sum_{i=1}^{\ell} |x_i|, \quad \|x\|_\infty = \max\{|x_i| : 1 \leq i \leq \ell\}$$

int A denotes the interior of the set A

$e \in R^\ell$ denotes $e = (1,1,\ldots,1)$

For any $A \subset R^\ell$, con A denotes the convex hull of A

$\setminus$ denotes the set theoretic subtraction

2^A denotes the set of all subsets of the set A

$|S|$ denotes the number of elements in the set S.

2.2. *Definitions*

The *commodity space* is the positive cone of R^ℓ, i.e., R_+^ℓ. A *utility function* u for a binary relation $\succcurlyeq$ on R_+^ℓ is a real valued function on R_+^ℓ which is order preserving, i.e., $u(x) \geq u(y) \Leftrightarrow x \succcurlyeq y$. Let U denote the space of all utility functions (assumed unique up to a linear affine transformation). A *cardinal finite exchange economy* is a map $\mathcal{E}: T \to U \times R_+^\ell$ where $T = \{1,2,\ldots,n\}$ is a finite set of agents and $U \times R_+^\ell$ is the space of agents' characteristics. For each $t \in T$, define $\mathcal{E}(t) = (u(t), e(t))$; $e(t)$ is the initial endowment of agent t and $u(t) \equiv u_t$ is his/her utility function. Denote by $\mathcal{P}$ the set of all binary relations $\succcurlyeq$ on R_+^ℓ. An *ordinal finite exchange economy* is a map $\mathcal{E}: T \to \mathcal{P} \times R_+^\ell$ where $\mathcal{P} \times R_+^\ell$ is the space of agents' characteristics. For each $t \in T$ define $\mathcal{E}(t) = (\succcurlyeq_t, e(t))$. We interpret $e(t)$ as the initial endowment of agent t and $\succcurlyeq_t$ as his/her preference relation. Obviously, any cardinal economy can be thought of as an ordinal economy simply by taking $\succcurlyeq_t$ to be the binary relation represented by u_t. An *assignment* x is a mapping of T into R_+^ℓ. An *allocation* x is an assignment such that $\sum_{t \in T} x(t) = \sum_{t \in T} e(t)$. A *game with side payments* $G = (T,V)$ consists of a finite set of agents T and a superadditive function $V:2^T \to R$ such that $V(\emptyset) = 0$. Each $S \in 2^T$ is a coalition and $V(S)$ expresses the "worth" of coalition S. The *Shapley value* of a game G is a rule which assigns to each agent a "payoff" equal to his expected marginal contribution to all possible coalitions according to the formula:

$$Sh_t = \sum_{S \subset T} \frac{(|S|-1)!(|T|-|S|)!}{|T|!} [V(S) - V(S \setminus \{t\})] \tag{2.1}$$

2.3. *Value Allocations*

To each cardinal finite exchange economy $\mathcal{E}:T \to U \times R_+^\ell$ and each vector of "weights" $\lambda \in R_+^{|T|}$ we may associate a game $G = (T,V_{\lambda u})$ according to the rule

$$V_{\lambda u}(S) = \max\{\sum_{t \in S} \lambda(t) u_t(x(t)): \sum_{t \in S} x(t) = \sum_{t \in S} e(t) \quad \text{for all}$$
$$x(t) \in R_+^\ell \quad \text{and all} \quad t \in S\}.$$

An allocation $x:T \to R_+^\ell$ is a *cardinal value allocation* for the economy $\mathcal{E}:T \to U \times R_+^\ell$, if for some set of "weights" $\lambda \in R_+^{|T|}$, $\lambda(t)u_t(x(t))$ is the Shapley value of the game $G = (T,V_{\lambda u})$ for each $t \in T$.

An allocation $x:T \to R_+^\ell$ is said to be an *ordinal value allocation* for the economy $\mathcal{E}:T \to \mathcal{P} \times R_+^\ell$ if there exists a family of utility functions $\{u_t\}_{t \in T}$ representing the preferences $\{\succcurlyeq_t\}_{t \in T}$ such that $u_t(x(t))$ is the Shapley value of the game $G = (T,V_u)$ for all $t \in T$. It is important to note that the ordinal value allocation depends on the representation of preferences.

Also note that at a value allocation the utility of each agent in the economy is given by summing up his/her dividends from the marginal value of each coalition in which he/she is a member. In that sense, it has been suggested in the literature that "equity" is inherent in the concept of value allocation. However, as we will discuss in Sections 4 and 5, the equitability of value allocations may be open to question.

2.4. *Competitive Equilibrium and Core*

A *price system* p is a vector in R_+^ℓ. The pair (p,x) where p is a price system and x is an allocation is said to be a *competitive* *equilibrium* for the economy $\mathcal{E}:T \to \mathcal{P} \times R_+^\ell$, if for each agent $t \in T$, $x(t)$ is maximal for $\succ_t$ in the budget set $B(p,t) = \{y \in R_+^\ell : p \cdot y \le p \cdot e(t)\}$. An assignment $x:T \to R_+$ is said to be *blocked* by a coalition S if there exists an assignment $y:T \to R_+^\ell$ such that $y(t) \succ_t x(t)$ for all $t \in S$ and $\sum_{t \in S} y(t) = \sum_{t \in S} e(t)$. The set of all allocations which cannot be blocked by any coalition constitutes the *core* of the economy $\mathcal{E}:T \to \mathcal{P} \times R_+^\ell$.

2.5. *Nondiscriminatory Allocations*

An allocation $x:T \to R_+^\ell$ is said to be *nondiscriminatory* if there exist no assignment $y:T \to R_+^\ell$ and disjoint coalitions S_1, S_2 such that for all $t \in S_1$, $y(t) \succ_t x(t)$ and $\sum_{t \in S_1} (y(t) - e(t)) = \sum_{t \in S_2} (x(t) - e(t))$. In other words, an allocation is said to be nondiscriminatory if no group of agents can redistribute among its members that net trade of any other group of agents and become better off. This notion of equity was introduced by Gabszewicz (1975). The concept of nondiscrimination, like the core, is cooperative. However, it is a sharper solution concept than the core. Indeed, as we will see, it turns out that all nondiscriminatory allocations are contained in the core, but the reverse is not true. Further, competitive equilibrium allocations are nondiscriminatory, but the reverse is not true. Consequently, the set of nondiscriminatory allocations is larger than the set of competitive equilibrium allocations, but it is smaller than the set of core allocations.

It is not clear whether the notion of nondiscrimination is informationally less demanding than a competitive equilibrium where each trader must know the price of each commodity in the market. Obviously, acquiring price information is not an easy task and may be very costly. On the other hand, the notion of nondiscrimination is based on coalitions of traders, and we do not know how easily traders can form coalitions. In that sense, it is not clear which notion requires less information. What is important to note is that the concept of nondiscrimination retains the appealing characteristics of both the competitive equilibrium and the core, i.e., it depends on preferences and endowments. (Contrast this with the cardinal value, where each agents' characteristics are preferences and endowments, but the weights of each agent are endogenously determined. See the discussion in Section 5.)

2.6. *The Relationship of the Above Solution Concepts*

Although all the above solution concepts coincide in an infinite economy, i.e., an economy where the set of agents is an atomless measure space, in a finite economy none of these solution concepts need be the same. Below we examine their relationship.

PROPOSITION 2.1. *An ordinal value allocation need not be in the core.*

PROOF. Consider an economy consisting of three agents $T = \{1,2,3\}$ and two goods x, y. Utility functions and initial endowments are given as follows:

$$u_1(x(1),y(1)) = \frac{x(1) + y(1)}{2} \quad , \; e(1) = (0,0)$$

$$u_2(x(2),y(2)) = \left(\frac{\sqrt{x(2)} + \sqrt{y(2)}}{2}\right)^2 , \; e(2) = (0,4)$$

$$u_3(x(3),y(3)) = \left(\frac{\sqrt{x(3)} + \sqrt{y(3)}}{2}\right)^2 , \; e(3) = (4,0) .$$

A computation of the characteristic function V_u gives:

$$V_u(\{1\}) = 0$$

$$V_u(\{2\}) = V_u(\{3\}) = 1$$

$$V_u(\{1,2\}) = V_u(\{1,3\}) = 2$$

$$V_u(\{2,3\}) = V_u(\{1,2,3\}) = 4 .$$

The Shapley values can be easily found to be $Sh_1 = \frac{2}{6}$, $Sh_2 = Sh_3 = \frac{11}{6}$. The ordinal value allocation is $(x(1), y(1)) = (\frac{2}{6},\frac{2}{6})$, $(x(2),y(2)) = (x(3), y(3)) = (\frac{11}{6},\frac{11}{6})$. Note that the above value allocation is not in the core, since the coalition $S = \{2,3\}$ can block it. This completes the proof of Proposition 2.1. ∎

PROPOSITION 2.2. *A cardinal value allocation need not be in the core.*

PROOF. It follows from the example used in the proof of Proposition 2.1 by letting $\lambda(1) = \lambda(2) = \lambda(3) = 1$. Indeed, for this choice of weights the value allocation in the ordinal sense is a value allocation in the cardinal sense as well. ∎

PROPOSITION 2.3. *An ordinal value allocation need not be a competitive equilibrium.*

PROOF. Since any competitive equilibrium is in the core (Debreu-Scarf (1963)) the result follows from Proposition 2.1. ■

PROPOSITION 2.4. *A cardinal value allocation need not be a competitive equilibrium.*

PROOF. Since all competitive equilibrium allocations are in the core the result follows from Proposition 2.2. ■

PROPOSITION 2.5. *A competitive equilibrium allocation need not be an ordinal value allocation.*

PROOF. See Aumann (1975), p. 634. ■

PROPOSITION 2.6. *A competitive equilibrium allocation need not be a cardinal value allocation.*

PROOF. Since any cardinal value allocation is an ordinal value allocation the result follows from Proposition 2.5. ■

PROPOSITION 2.7. *Any competitive equilibrium is nondiscriminatory.*

PROOF. Let (p,x) be a competitive equilibrium and assume that x is not nondiscriminatory. Then there exist an assignment $y:T \to R_+^{\ell}$ and disjoint coalitions S_1, S_2 such that $y(t) \succ_t x(t)$ for all $t \in S_1$ and

$$\sum_{t\in S_1}(y(t) - e(t)) = \sum_{t\in S_2}(x(t) - e(t)) \; . \tag{2.2}$$

But, $y(t) \succ_t x(t)$ for all $t \in S_1$ implies that $p\cdot y(t) > p\cdot e(t)$ for all $t \in S_1$ and consequently

$$p \cdot \sum_{t\in S_1} y(t) > p \cdot \sum_{t\in S_1} e(t) \tag{2.3}$$

From (2.2) and (2.3) it follows that $p\cdot \sum_{t\in S_2} (x(t) - e(t)) > 0$ or $p\cdot \sum_{t\in S_2} x(t) > p\cdot \sum_{t\in S_2} e(t)$, a contradiction to the fact that (x,p) is a competitive equilibrium. ■

PROPOSITION 2.8. *Any nondiscriminatory allocation is in the core.*

PROOF. Let $x:T \to R_+^\ell$ be a nondiscriminatory allocation and suppose that x is not a core allocation. Then there exist a coalition S_1 and an assignment $y:T \to R_+^\ell$ such that $y(t) \succ_t x(t)$ for all $t \in S_1$, and $\sum_{t \in S_1} y(t) = \sum_{t \in S_1} e(t)$. Let S_2 be the empty coalition. Then $S_1 \cap S_2 = \emptyset$ and $\sum_{t \in S_1} (y(t) - e(t)) = \sum_{t \in S_2} (x(t) - e(t))$, a contradiction to the fact that $x:T \to R_+^\ell$ is a nondiscriminatory allocation. ∎

PROPOSITION 2.9. *An ordinal value allocation need not be nondiscriminatory.*

PROOF. It follows from Proposition 2.1 and 2.8. ∎

PROPOSITION 2.10. *A cardinal value allocation need not be nondiscriminatory.*

PROOF. It follows from Propositions 2.2 and 2.8. ∎

PROPOSITION 2.11. *(a) A core allocation need not be nondiscriminatory and (b) a nondiscriminatory allocation need not be competitive.*

PROOF. The proof is by means of an example due to Gabszewicz (1975). Consider an economy with three agents $T = \{1,2,3\}$ and two goods x, y. Agents' utility functions and initial endowments are:

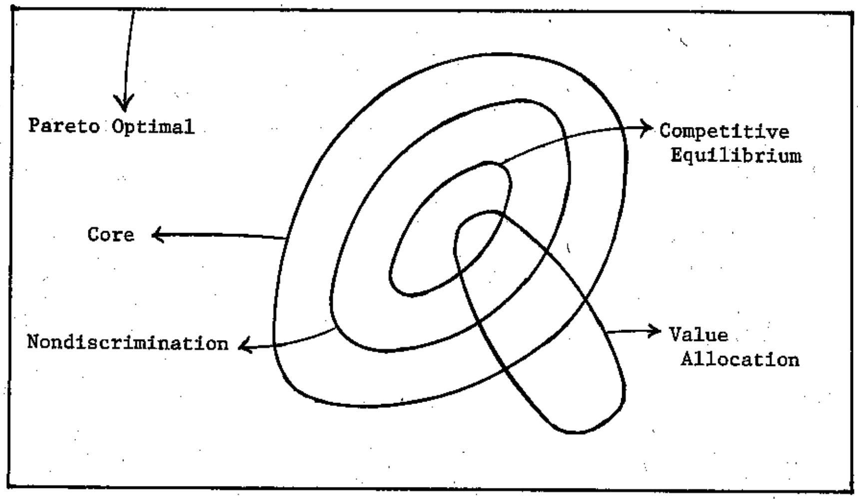
$$u_i(x(i), y(i)) = \sqrt{x(i)} + \sqrt{y(i)}, \qquad i = 1,2,3$$

$$e(1) = (0,8),\ e(2) = e(3) = (4,0).$$

(a) The allocation $(x(1), y(1)) = (5.5,5.5)$, $(x(2),y(2)) = (1.5,1.5)$, $(x(3),y(3)) = (1,1)$ is in the core. However, for $S_1 = \{3\}$ and $S_2 = \{2\}$ it can be easily checked that $x:T \to R_+^\ell$ is not nondiscriminatory.

(b) The allocation $(x(1),y(1)) = (6,6)$, $(x(2),y(2)) = (x(3), y(3)) = (1,1)$ is nondiscriminatory but it is easily seen that it is not competitive. ∎

The relationship of the above solution concepts can be depicted in the following diagram.



It is clear from the diagram above that among the concepts of core, competitive equilibrium and nondiscrimination the competitive equilibrium is the sharpest as it is contained in the other two. However, among value allocations and the other three concepts we cannot determine whether or not the value is sharper than the other solution concepts. Nevertheless, in the next section, we will try to point out several properties that value allocations may have, and compare this concept with the competitive equilibrium.

Recently Bennet (1983) and Bennet-Zame (1983) have introduced a new solution concept for Non-Transferable Utility (NTU) games called bargaining aspirations. In a series of examples the authors show that the concept of bargaining aspirations provides reasonable and sensible results. This concept seems very promising in applications to exchange economies.

3. MANIPULABILITY OF VALUE

It is known (Hurwicz (1972) and Thomson (1979)) that the competitive equilibrium is manipulable in the sense that one agent can misrepresent his/her preferences and become better off. Recently, Thomson (1983) examined the manipulability of value in transferable utility economies. Specifically utility functions were additive, separable and linear in one commodity. Here we will show that for a more general class of economies value allocations (either cardinal or ordinal) are susceptible to misrepresentation of preferences.

We will first need some notation and definitions. Denote by F the set of all attainable allocations, i.e., $F = \{x \in R_+^{\ell|T|} : \sum_{t\in T} x(t) = \sum_{t\in T} e(t)\}$, and by $V_u(\mathcal{E})$ and $V_{\lambda u}(\mathcal{E})$ the sets of all ordinal and cardinal value allocations respectively, i.e., $z \in V_u(\mathcal{E}) \Leftrightarrow z \in F$ and $u_t(z(t)) = Sh_t$ for all $t \in T$ and $z \in V_{\lambda u}(\mathcal{E}) \Leftrightarrow z \in F$ and $\lambda(t)u_t(z(t)) = Sh_t$ for all $t \in T$. Denote by E the set of all economies.

An *ordinal value allocation mechanism* is a mapping ϕ on E which redistributes the endowment in the economy according to the Shapley value rule, i.e., $\phi(\mathcal{E}) = \{x \in R_+^{\ell|T|} : x \in V_u(\mathcal{E})\}$. Similarly one can define a *cardinal value allocation mechanism* as a mapping ψ defined on E by $\psi(\mathcal{E}) = \{x \in R_+^{\ell|T|} : x \in V_{\lambda u}(\mathcal{E})\}$.

An ordinal value allocation mechanism ϕ is *manipulable* if there exist $\mathcal{E} = \{(u(1),e(1)),\dots,(u(n),e(n))\}$, $\mathcal{E}' = \{(u'(1),e(1)),\dots(u'(n),e(n))\}$ in E and $x \in \phi(\mathcal{E})$, $x' \in \phi(\mathcal{E}')$ with

(i) $u_s' \neq u_s$ for some s in T, $u_t' = u_t$ for $t \neq s$ and

(ii) $u_s(x'(s)) > u_s(x(s))$.

A cardinal value allocation mechanism ψ is *manipulable* , if there exist $\mathcal{E} = \{(u(1),e(1)),\dots,(u(n),e(n))\}$, $\mathcal{E}' = \{(u'(1),e(1)),\dots,(u'(n),e(n))\}$ in E and $x \in \psi(\mathcal{E})$, $x' \in \psi(\mathcal{E}')$ with

(i) $u_s' \neq u_s$ for some s in T, $u_t' = u_t$ for $t \neq s$ and

(ii) $u_s(x'(s)) > u_s(x(s))$.

In other words a value allocation mechanism either in the ordinal or cardinal sense, is manipulable in some economy $\mathcal{E}$, if at least one agent will misrepresent his/her preferences and make the economy to appear as $\mathcal{E}'$. In this new economy the agent who misrepresented his/her preferences is now better off.

THEOREM 3.1. *The ordinal value allocation mechanism is manipulable.*

PROOF. The theorem is proved by means of an example. Consider an economy with three agents, $T = \{1,2,3\}$ and two goods, x, y. Agents' utility functions and initial endowments are given as follows:

$$u_1(x(1),y(1)) = \sqrt{x(1)y(1)} \tag{3.1}$$

$$u_2(x(2),y(2)) = \sqrt{x(2)y(2)} \tag{3.2}$$

$$u_3(x(3),y(3)) = \frac{x(3) + y(3)}{2} \tag{3.3}$$

$$e(1) = (0,4) \tag{3.4}$$

$$e(2) = (4,0) \tag{3.5}$$

$$e(3) = (0,0). \tag{3.6}$$

A computation of the characteristic function V_u gives:

$$V_u(\{1\}) = V_u(\{2\}) = V_u(\{3\}) = 0$$

$$V_u(\{1,3\}) = V_u(\{2,3\}) = 2$$

$$V_u(\{1,2\}) = V_u(\{1,2,3\}) = 4.$$

We can compute the Shapley values of each agent and find that $Sh_1 = Sh_2 = \frac{10}{6}$, $Sh_3 = \frac{4}{6}$. It can be easily verified that the ordinal value allocation is $(x(1),y(1)) = (x(2),y(2)) = (\frac{10}{6},\frac{10}{6})$, $(x(3),y(3)) = (\frac{4}{6},\frac{4}{6})$.

Suppose now that agent 1 does not report his/her true preferences. Instead, he/she misrepresents his/her self by reporting a false utility function, i.e.,

$$u_1'(x(1),y(1)) = \frac{x(1) + y(1)}{2}. \tag{3.7}$$

Obviously (3.2)-(3.6) remain the same. Therefore we have now a new economy $\mathcal{E}'$ and we can again compute the characteristic function V_u and find:

$$V_u(\{1\}) = 2$$

$$V_u(\{2\}) = V_u(\{3\}) = 0$$

$V_u(\{1,3\}) = V_u(\{2,3\}) = 2$

$V_u(\{1,2\}) = V_u(\{1,2,3\}) = 4.$

The Shapley values are $Sh_1 = \frac{14}{6}$, $Sh_2 = \frac{8}{6}$, $Sh_3 = \frac{2}{6}$. The ordinal value allocation for the new economy $\mathcal{E}'$ is $(x'(1),y'(1)) = (\frac{14}{6},\frac{14}{6})$, $(x'(2),y'(2)) = (\frac{8}{6},\frac{8}{6})$, $(x'(3),y'(3)) = (\frac{4}{6},\frac{4}{6})$. Note that agent 1 is now better off, i.e., $u_1(x'(1),y'(1)) = u_1(\frac{14}{6},\frac{14}{6}) = \frac{14}{6} > u_1(x(1),y(1)) = u_1(\frac{10}{6},\frac{10}{6}) = \frac{10}{6}$, and necessarily agents 2 and 3 are worse off. This completes the proof of the theorem. ∎

THEOREM 3.2. *The cardinal value allocation mechanism is manipulable.*

PROOF. It follows from Theorem 3.1 by letting $\lambda(1) = \lambda(2) = \lambda(3) = 1$. Indeed, for the above choice of weights the ordinal value allocation in the example used in the proof of Theorem 3.1 is also a cardinal value. ∎

4. VALUE AND FAIRNESS

An allocation $x:T \to R_+^\ell$ is said to be *Pareto optimal* if there is no allocation $x':T \to R_+^\ell$ such that $u_t(x'(t)) \geq u_t(x(t))$ for all $t \in T$ and $u_t(x'(t)) > u_t(x(t))$ for at least one t. An assignment $x:T \to R_+^\ell$ is said to be *fair* (or *envy free*) if it is Pareto optimal and $u_t(x(t)) \geq u_t(x(s))$ for all t, s in T. If $u_t(x(t)) < u_t(x(s))$ we will say that agent t *envies* agent s at the assignment x.

Although it is hardly reasonable to require a mechanism to be fair from any arbitrary position (in fact Feldman-Kirman (1974) show that the competitive equilibrium may fail to have this property), it is known (see Varian (1974) or Thomson-Varian (1984)) that the competitive equilibrium treats agents with equal income or identical initial endowments in a fair way. Unfortunately as we will see, value allocations either in the cardinal or ordinal sense need not treat agents with identical initial endowments in a non-envy free manner.

PROPOSITION 4.1. *An ordinal value allocation need not treat agents with identical initial endowments in a fair way.*

PROOF. Consider an economy consisting of four agents $T = \{1,2,3,4\}$ and two commodities x, y. The utility function and initial endowment of each agent is given as follows:

$$u_1(x(1),y(1)) = 2x(1) + 2y(1), \qquad e(1) = (2,2)$$

$$u_2(x(2),y(2)) = (\sqrt{x(2)} + \sqrt{y(2)})^2, \qquad e(2) = (2,2)$$

$$u_3(x(3),y(3)) = (\sqrt{x(3)} + \sqrt{y(3)})^2, \qquad e(3) = (0,2)$$

$$u_4(x(4),y(4)) = (\sqrt{x(4)} + \sqrt{y(4)})^2, \qquad e(4) = (2,0) \ .$$

Note that agents 1 and 2 have identical initial endowments. However, as we will see agent 2 envies agent 1's value allocation. A computation of the characteristic function V_u gives:

$$V_u(\{1\}) = V_u(\{2\}) = 8$$

$$V_u(\{1,2\}) = 16$$

$$V_u(\{3\}) = V_u(\{4\}) = 2$$

$$V_u(\{1,3\}) = V_u(\{1,4\}) = 12$$

$$V_u(\{2,3\}) = V_u(\{2,4\}) = 6 + 4\sqrt{2}$$

$$V_u(\{3,4\}) = 8$$

$$V_u(\{1,2,3\}) = V_u(\{1,2,4\}) = 20$$

$$V_u(\{1,3,4\}) = 16$$

$$V_u(\{2,3,4\}) = 16$$

$$V_u(\{1,2,3,4\}) = 24 \ .$$

We can also compute the Shapley value according to formula (2.1) and find that $Sh_1 = \frac{224 - 16\sqrt{2}}{24}$, $Sh_2 = \frac{176 + 16\sqrt{2}}{24}$, $Sh_3 = Sh_4 = \frac{176}{48}$. It can be easily verified that the ordinal value allocation is:

$$(x(1),y(1)) = \left(\frac{224 - 16\sqrt{2}}{96}, \frac{224 - 16\sqrt{2}}{96}\right),$$

$$(x(2),y(2)) = \left(\frac{176 + 16\sqrt{2}}{96}, \frac{176 + 16\sqrt{2}}{96}\right),$$

$$(x(3),y(3)) = (x(4),y(4)) = \left(\frac{176}{192}, \frac{176}{192}\right).$$

However, at the above ordinal value allocation agent 2 envies agent 1's assignment since $u_2(x(1),y(1)) = u_2\left(\frac{224 - 16\sqrt{2}}{96}, \frac{224 - 16\sqrt{2}}{96}\right) = \frac{224 - 16\sqrt{2}}{24} > u_2(x(2),y(2)) = u_2\left(\frac{176 + 16\sqrt{2}}{96}, \frac{176 + 16\sqrt{2}}{96}\right) = \frac{176 + 16\sqrt{2}}{24}$.

Hence, the above ordinal value allocation stemming from the equal division of initial endowments for agents 1 and 2 does not treat them in a fair way. This completes the proof of Proposition 4.1. ∎

PROPOSITION 4.2. *A cardinal value allocation need not treat agents with identical initial endowments in a fair way.*

PROOF. It follows from Proposition 4.1 by letting $\lambda(1) = \lambda(2) = \lambda(3) = \lambda(4) = 1$. ∎

Propositions 4.1 and 4.2 cast doubt on the view that value allocations are more "equitable" than competitive equilibria. In fact, these results suggest that they are not. We obtain these conclusions in Proposition 4.1 because agent 1 has a superior "utility producing technology" and consequently increases the characteristic function V_u of any coalition of which he/she is a member. In particular, observe that $2x + 2y > (\sqrt{x} + \sqrt{y})^2$ for $x \neq y$ and with equality if $x = y$. Hence since utility is transferable, it is optimal for agents 2, 3, and 4 to transfer their initial endowments to agent 1 and increase $V_u(S)$, $S \subset T$. Consequently, the expected marginal contribution of agent 1 to all the coalitions of which he/she is a member, i.e., his/her Shapley value, becomes higher than agent 2's even though they both start out with identical initial endowments.

Here we must note that the core of an economy may give rise to a similar pathology. In fact Feldman-Kirman (1974) show that the core from the equal division of initial endowments need not be fair.

5. VALUE AND THE EQUAL TREATMENT PROPERTY

We will say that two agents in an economy are *identical* if they have the same characteristics, i.e., two agents t_1 and t_2 are identical if $\mathcal{E}(t_1) = (u(t_1), e(t_1)) = \mathcal{E}(t_2) = (u(t_2), e(t_2))$. It is known that some solution concepts (for example the competitive equilibrium or Nash solution) treat identical agents the same way. This is certainly a desirable property that we would like any solution concept to have. Unfortunately, we will show that the cardinal value allocation fails to have this property. Before we state our result we will need the following definition. A cardinal value allocation mechanism ψ is *symmetric* (or it has the *equal treatment property*) if for every $\mathcal{E} = \{(u(1),e(1)),\ldots,(u(n),e(n))\}$ in E, for every $x \in V_{\lambda u}(\mathcal{E})$ and for every pair (t_1,t_2) of identical agents we have that $u(x(t_1),t_1) = u(x(t_2),t_2)$.

PROPOSITION 5.1. *The cardinal value allocation mechanism* ψ *need not be symmetric.*

PROOF. The proof is by means of an example similar to that of Shafer (1980) and Scafuri-Yannelis (1984). Consider an exchange economy with four agents and two commodities denoted by $T = \{1,2,3,4\}$ and x, y, respectively. Agents' utility functions and initial endowments are given as follows:

$$u_1(x(1),y(1)) = \left(\tfrac{1}{2}\sqrt{x(1)} + \tfrac{1}{2}\sqrt{y(1)}\right)^2, \qquad e(1) = (0,1)$$

$$u_2(x(2),y(2)) = \left(\tfrac{1}{2}\sqrt{x(2)} + \tfrac{1}{2}\sqrt{y(2)}\right)^2, \qquad e(2) = (1,0)$$

$$u_3(x(3),y(3)) = \tfrac{1}{2}x(3) + \tfrac{1}{2}y(3), \qquad e(3) = (0,0)$$

$$u_4(x(4),y(4)) = \tfrac{1}{2}x(4) + \tfrac{1}{2}y(4), \qquad e(4) = (0,0).$$

Agents' weights are $\lambda(1) = \lambda(2) = \lambda(3) = 1$ and $\lambda(4) = \frac{1}{4}$. Note that agents 3 and 4 are identical, i.e., have the same utility functions and the same initial endowments. A computation of the characteristic function $V_{\lambda u}$ gives:

$$V_{\lambda u}(\{1\}) = V_{\lambda u}(\{2\}) = \tfrac{1}{4}$$

$$V_{\lambda u}(\{3\}) = V_{\lambda u}(\{4\}) = V_{\lambda u}(\{3,4\}) = 0$$

$$V_{\lambda u}(\{2,4\}) = V_{\lambda u}(\{1,4\}) = \frac{1}{4}$$

$$V_{\lambda u}(\{2,3\}) = V_{\lambda u}(\{1,3\}) = \frac{1}{2}$$

$$V_{\lambda u}(\{1,2\}) = 1$$

$$V_{\lambda u}(\{2,3,4\}) = V_{\lambda u}(\{1,3,4\}) = \frac{1}{2}$$

$$V_{\lambda u}(\{1,2,4\}) = V_{\lambda u}(\{1,2,3\}) = V_{\lambda u}(\{1,2,3,4\}) = 1.$$

Applying formula (2.1) we can compute the Shapley values of agents 1, 2, 3, 4 and find that $Sh_1 = \frac{11}{24}$, $Sh_2 = \frac{11}{24}$, $Sh_3 = \frac{2}{24}$, $Sh_4 = 0$. The cardinal value allocation is $((x(1),y(1)) = \left(\frac{11}{24},\frac{11}{24}\right)$, $(x(2),y(2)) = \left(\frac{11}{24},\frac{11}{24}\right)$, $(x(3),y(3)) = \left(\frac{2}{24},\frac{2}{24}\right)$, $(x(4),y(4)) = (0,0)$. However, at the above cardinal value allocation agents 3 and 4, who are identical, are treated very differently with $u_3(x(3),y(3)) = \frac{2}{24} \neq u_4(x(4),y(4)) = 0$. Hence, the cardinal value allocation mechanism need not be symmetric and this completes the proof. ∎

The asymmetry of the cardinal value allocation is a consequence of the peculiarity of the concept itself. Specifically, on the one hand each agent in the economy is characterized by his preferences and initial endowments, and on the other hand the "weights" of each agents are endogenously determined (the fixed point provides the weights). Consequently, even if two agents are identical in the economy the fixed point may assign to them different weights, and therefore the resulting utility that they will derive at the cardinal value allocation need not be the same. In that sense, Proposition 5.1, casts doubt on any interpretation of the weights as a meaningful "endogenous utility comparison" as has been suggested in Shapley (1969). Contrary to suggestions in the literature, Proposition 5.1 appears to indicate that the cardinal value allocation is not an equitable solution concept. Finally, we should point out that we do not know whether or not the asymmetry of cardinal value allocations disappears as the economy gets large. Champsaur (1975), Mas-Colell (1977) and Cheng (1981) have shown that the set of cardinal value allocations converges to the set of competitive equilibrium allocations as the number of agents in the economy goes to infinity, if a symmetry requirement is satisfied. It remains an open question whether the cardinal value convergence theorem remains valid for allocations without the equal treatment property.

6. FAIRNESS IN MIXED ECONOMIES

We will now examine the previous solution concepts in economies where agents are characterized not only by their initial endowments and preferences, but also by their weight. Specifically, we will have two kinds of agents, "small" and "large". "Small" agents are those who have a small weight and therefore their impact on the market is small, (they may be thought of as price takers), and "large" agents are those whose weight is big and consequently they have a noticable impact on the market (one may think of a "large" agent as a monopolist or oligopolist). We will need the following definitions.

6.1. *Definitions*

Let $\mathcal{P}$ denote the set of binary relations $\succ$ on R_+^ℓ which may possess the following properties:

(i) transitivity: $x \succ y,\ y \succ z \Rightarrow x \succ z$.

(ii) irreflexivity: $x \nsucc x$.

(iii) continuity: $\{(x,y): x \succ y\}$ is relatively open.

(iv) weak transitivity: $x >> y$ and $y \succ z \Rightarrow x \succ z$.

(v) monotonicity: $x > y \Rightarrow x \succ y$.

(vi) convexity: for all $y \in R_+^\ell$ the set $\{x \in R_+^\ell : x \succ y\}$ is convex.

A *finite mixed exchange economy* $\mathcal{E}$ is a mapping of T into $\mathcal{P} \times R_+^\ell$, where T is a finite set of traders. Let $\succ_t$ be the projection of $\mathcal{E}(t)$ onto $\mathcal{P}$, and $e(t)$ the projection of $\mathcal{E}(t)$ onto R_+^ℓ. We interpret $\succ_t$ as the preference of trader t and $e(t)$ as his/her initial endowment. Partition the set of traders T into T_0 and T_1. We assume for all $t \in T_0$, $\mu(t) = \frac{1}{|T_0|}$ where $|T_0| = n$ and for all $t \in T_1$, $\mu(t) = 1 - \frac{1}{n}$. We name the traders in the set T_0 small and the traders in the set T_1 large. Hence the set of traders can be described by a discrete measure space (T,μ). An *allocation* is a function f of T into R_+^ℓ such that $\sum_{t\in T} f(t)\mu(t) = \sum_{t\in T} e(t)\mu(t)$. In this framework each trader has a specific weight, and an allocation redistributes initial endowments to traders according to their weights. In other words, the actual initial endowment of a trader is $e(t)$ multiplied by his weight $\mu(t)$, i.e., $e(t)\mu(t)$. Further, what each trader actually consumes is $f(t)\mu(t)$. However, since the preference relation $\succ_t$ of each trader is defined on consumption bundles $f(t)$ and not on actual consumption bundles $f(t)\mu(t)$, the fact that $f(t)$ is preferred to $y(t)$ will be assumed to be equivalent to $f(t)\mu(t)$ is preferred to $y(t)\mu(t)$.

The mixed economy just described is the finite analogue of an exchange economy whose set of traders is a measure space containing atoms and an atomless part. Moreover, if the set of large traders is empty, then we have the standard Arrow-Debreu-McKenzie exchange economy. Consequently, all the results to be proved in this section are true for the conventional Arrow-Debreu-McKenzie exchange economy.

Let $\mathcal{S}_0 = \{S:S \subseteq T_0\}$ and $\mathcal{S}_1 = \{S:T_1 \supseteq S\}$ be classes of coalitions of small and large traders respectively. An allocation $f:T \to R_+^\ell$ is said to be *nondiscriminatory relative to* $\mathcal{S}_0$ *and* $\mathcal{S}_1$ if there exist no disjoint coalitions $S_1 \in \mathcal{S}_0$ and $S_2 \in \mathcal{S}_1$ and assignement $y(t)$ such that $y(t) \succ_t f(t)$ for all $t \in S_i$, $i = 1,2$ and $\sum_{t\in S_i}(y(t)\mu(t) - e(t)\mu(t)) = \sum_{t\in S_j}(f(t)\mu(t) - e(t)\mu(t))$, $i \neq j$. In other words, an allocation is nondiscriminatory relative to $\mathcal{S}_0$ and $\mathcal{S}_1$ if no coalition of small (large) traders can redistribute among its members the net trade of any other coalition of large (small) traders and become better off. Denote by $C(\bar{\mathcal{E}})$ the *set of all core allocations* in $\bar{\mathcal{E}}$, and by $N(\bar{\mathcal{E}})$ the *set of all nondiscriminatory alocations relative to* $\mathcal{S}_0$ *and* $\mathcal{S}_1$ in $\bar{\mathcal{E}}$. Let $T_1 = \emptyset$, then an allocation f is said to be *nondiscrimatory* if there exist no disjoint coalitions $S_1 \in 2^{T_0}$ and $S_2 \in 2^{T_0}$ and $y:T \to R_+^\ell$ such that $y(t) \succ_t f(t)$ for all $t \in S_1$, and $\sum_{t\in S_1}(y(t)\mu(t) - e(t)\mu(t)) = \sum_{t\in S_2}(f(t)\mu(t) - e(t)\mu(t))$.

6.2. *Theorems*

THEOREM 6.1. *Let* $\bar{\mathcal{E}}:T \to \mathcal{P} \times R_+^\ell$ *be a finite mixed exchange economy. If* $\succ_t \in \mathcal{P}$ *satisfies (i), (ii), (iii), (v), (vi) and* $e(t) \gg 0$ *for all* $t \in T$, *then* $N(\bar{\mathcal{E}}) \neq \emptyset$.

THEOREM 6.2. *Let* $\mathcal{E}:T_0 \to \mathcal{P} \times R_+^\ell$ *be a finite exchange economy. Let* $\succ_t \in \mathcal{P}$ *satisfy (iv) and (v) for all* $t \in T_0$. *If* $f:T_0 \to R_+^\ell$ *is a nondiscriminatory allocation in* $\mathcal{E}$, *then there exists* $p \in P = \{q \in R_+^\ell: \sum_{i=1}^{\ell} q^i = 1\}$, *such that*

(i) $\sum_{t\in T_0} |p\cdot(f(t)\mu(t) - e(t)\mu(t))| \le \frac{2M}{n}$,

(ii) $\sum_{t\in T_0} |\inf\{p\cdot x(t)\mu(t) - p\cdot e(t)\mu(t) : x(t) \succ_t f(t)\}| \le \frac{2M}{n}$, *where*

$$M = \sup\{\|e(t_1) + \ldots + e(t_\ell)\|_\infty : t_i \in T_0,\ i = 1,\ldots,\ell\}.$$

THEOREM 6.3. *Let* $\bar{\mathcal{E}}:T \to \mathcal{P} \times R_+^\ell$ *be a finite mixed exchange economy. Let* $\succ_t \in \mathcal{P}$ *satisfy (i)-(iii) for all* $t \in T$ *and (vi) for all* $t \in T_1$. *Then there exists an approximate nondiscriminatory allocation* f *in* $\bar{\mathcal{E}}$, *i.e., there are no assignment* $y:T \to R_+^\ell$ *and disjoint coalitions* $S_1 \in \mathcal{S}_0$ *and* $S_2 \in \mathcal{S}_1$ *such that* $y(t) \succ_t f(t)$ *for all* $t \in S_i$, $i = 1,2$, $\sum_{t\in S_i} [y(t)\mu(t) - e(t)\mu(t)] = \sum_{t\in S_j} [f(t)\mu(t) - e(t)\mu(t)]$, $i \ne j$ *and* $\sum_{i=1}^{\ell} \max\{\sum_{t\in T} [f_i(t)\mu(t) - e_i(t)\mu(t)], 0\} \le \frac{1}{\sqrt{n}} (\ell+1) \max_{t\in T} \|e(t)\|$.

Theorem 6.1 provides sufficient conditions which guarantee the existence of nondiscriminatory allocations with respect to small and large traders. Note that the assumptions adopted in the above theorem are those needed for the existence of a competitive equilibrium in a finite economy with convex preferences.

Theorem 6.2 is absent of large traders and provides computable bounds on the degree of noncompetitiveness of nondiscriminatory allocations. Since competitive equilibrium allocations are nondiscriminatory, then Theorem 6.2 is an equivalence result. In particular, it gives a computable error under which nondiscriminatory allocations can be decentralized by an appropriately chosen price vector.

Theorem 6.3 proves the existence of approximate nondiscriminatory allocations relative to coalitions of small and large traders. Contrary to Theorem 6.1, the preferences of all small traders in Theorem 6.3 need not be convex and consequently exact nondiscriminatory allocations may not exist.

We must remark, that by allowing the number of small agents to increase indefinitely one may derive results for sequences of finite economies directly from Theorems 6.1-6.3. In fact, the sequential formulation captures precisely the meaning of small and large traders. Since for all $t \in T_0$, $\mu(t) = \frac{1}{n}$, then as the number of small traders, i.e., n, goes to infinity, the weight of each small trader goes to zero. Obviously, the weight of each large trader, i.e., $1 - \frac{1}{n}$, tends to one as n goes to infinity. Since the actual initial endowment assigned to each small trader is $\frac{e(t)}{n}$, then as n increases indefinitely, each small trader becomes infinitesimal, i.e., $\frac{e(t)}{n} \to 0$ as $n \to \infty$. On the other hand, the actual initial endowment of a large trader, i.e., $e(t)(1 - \frac{1}{n})$ tends to $e(t)$ as $n \to \infty$. Consequently, a sequence of finite mixed exchange economies provides nice interpretations of the modeling of small and large traders since, as the economy gets large, the former's impact becomes infinitesimal and the latter's noninfinitesimal. Clearly, such a sequence of finite mixed exchange economies corresponds to an exchange economy where the set of traders is a measure space containing atoms and an atomless part. In this framework small traders are identified with the atomless part of the measure space and large traders with the atomic.

Finally, by imposing the condition that the initial endowments of small agents are integrable, i.e., no "small" group of small agents can hold sufficiently large endowments one may immediately derive sequential counterparts from Theorems 6.1-6.3.

6.3. Proofs

PROOF OF THEOREM 6.1. We will show that the set of competitive equilibrium allocations denoted by $CE(\bar{\mathcal{E}})$ is a subset of $N(\bar{\mathcal{E}})$. Let $f \in CE(\bar{\mathcal{E}})$, i.e., for all $t \in T$, $p \cdot f(t) \le p \cdot e(t)$, $f(t)$ is maximal for $\succ_t$ in $\{x \in R_+^\ell : p \cdot x \le p \cdot e(t)\}$ and $\sum_{t \in T} f(t)\mu(t) = \sum_{t \in T} e(t)\mu(t)$. Suppose that $f \notin N(\bar{\mathcal{E}})$. Then there exist S_1, S_2, $S_1 \cap S_2 = \emptyset$, $S_1 \in \mathcal{S}_0$, $S_2 \in \mathcal{S}_1$ and $y(t)$ such that $y(t) \succ_t f(t)$ for all $t \in S_1$ and

$$\sum_{t \in S_1} (y(t)\mu(t) - e(t)\mu(t)) = \sum_{t \in S_2} (f(t)\mu(t) - e(t)\mu(t)) . \qquad (6.1)$$

But, $y(t) \succ_t f(t)$ for all $t \in S_1$ implies $p \cdot y(t) > p \cdot e(t)$ for all $t \in S_1$ and consequently,

$$p \cdot \sum_{t \in S_1} y(t)\mu(t) > p \cdot \sum_{t \in S_1} e(t)\mu(t) . \qquad (6.2)$$

From (6.1) and (6.2) it follows that $p \cdot \sum_{t \in S_2} (f(t)\mu(t) - e(t)\mu(t)) > 0$ or $p \cdot \sum_{t \in S_2} f(t) > p \cdot \sum_{t \in S_2} e(t)$, a contradiction to the fact that $f \in CE(\bar{\mathcal{E}})$. Suppose now that $S_1 \in \mathcal{S}_1$ and $S_2 \in \mathcal{S}_0$, and repeat exactly the same argument with the above to reach another contradiction. Hence we conclude that $CE(\bar{\mathcal{E}}) \subset N(\bar{\mathcal{E}})$. Since the assumptions of Theorem 6.1 guarantee that $CE(\bar{\mathcal{E}}) \ne \emptyset$, then $N(\bar{\mathcal{E}}) \ne \emptyset$. ∎

We will now need the following proposition which extends a result of Anderson-Khan-Rashid (1983) into exchange economies with small and large traders. The proof is patterned after theirs.

Let p be a price vector in R^{ℓ}_{+}; define, as in Anderson-Khan-Rashid (1983), the excess demand set of trader t by $D(p,t) = \{z - e(t) : z \in R^{\ell}_{+},\ p\cdot z \le p\cdot e(t),\ y \succ_t z \Rightarrow p\cdot y > p\cdot e(t)\}$. Let

$$\Delta = \{p \in R^{\ell} :\ 0 < p_i \leqslant 1 \text{ for } i = 1,\ldots,\ell\}.$$

PROPOSITION 6.1. *Let* $\mathcal{E}: T \to \mathcal{P} \times R^{\ell}_{+}$ *be a finite mixed exchange economy. Let* $\succ_t \in \mathcal{P}$ *satisfy* (i)-(iii) *for all* $t \in T$ *and* (vi) *for all* $t \in T_1$. *Then there exist* $x: T \to R^{\ell}_{+}$ *and* $p \in \Delta$ *such that* $x(t) \in D(p,t)$ *for all* $t \in T$ *and* $\sum_{i=1}^{\ell} \max\{\sum_{t\in T} x_i(t)\mu(t), 0\} \le \frac{1}{\sqrt{n}}(\ell + 1)\max_{t\in T}\|e(t)\|$.

PROOF. Let $\zeta(p) = \sum_{t\in T} D(p,t)\mu(t)$

$$Z = \{z \in R^{\ell}_{+} :\ \|z\| \le \max_{t\in T}\|e(t)\|\}$$

$$\Delta' = \{p \in \Delta : p_i \ge \frac{1}{\sqrt{n}},\ i = 1,\ldots,\ell\}$$

$$\phi(z) = \{p \in \Delta' : p\cdot z \ge q\cdot z \text{ for all } q \in \Delta\}$$

$$\theta(p,z) = \phi(z) \times \text{con}\ \zeta(p).$$

Clearly, θ is a set valued function from the set $\Delta' \times Z$ into itself. A standard argument can be applied to show that the mapping θ satisfies all the properties of the Kakutani fixed point theorem. Consequently, there exist $(p,z) \in \theta(p,z)$, i.e., $z \in \text{con}\ \zeta(p)$ and for all $z \in \Delta$, $q\cdot z \le p\cdot z \le 0$. By the Shapley-Folkman theorem we can write $z = \sum_{t\in T'} z(t)\mu(t) + \sum_{j=1}^{\ell} y(t_j)\mu(t_j)$ where $T' = T_1 \cup (T_0\setminus\{t_1,\ldots,t_\ell\})$, and $z(t) \in D(p,t)$ for all $t \in T'$ and $y(t_j) \in \text{con}\ D(p,t_j)$, $j = 1, \ldots, \ell$. Pick arbitrarily $z(t_j) \in D(p,t_j)$, $j = 1,\ldots,\ell$. Then, $z(t_j) - y(t_j) = z(t_j) + e(t_j) - (y(t_j) + e(t_j)) \le z(t_j) + e(t_j)$. Hence, $\sum_{i=1}^{\ell} \max\{(z_i(t_j) - y_i(t_j))\mu(t_j), 0\} \le \sum_{i=1}^{\ell} (z_i(t_j) + e_i(t_j))\mu(t_j)$. It follows from the definition of p and the budget set that $\sum_{i=1}^{\ell}(z_i(t_j) + e_i(t_j))\mu(t_j) \le \frac{p\cdot e(t_j)\mu(t_j)}{\min(p_1,\ldots,p_\ell)} \le \frac{1}{\sqrt{n}}\|e(t_j)\|$. Thus,

$q\cdot\sum_{t\in T} z(t)\mu(t) = q\cdot z + q\cdot(\sum_{j=1}^{\ell}(z(t_j) - y(t_j))\mu(t_j)) \leqslant \frac{\ell}{\sqrt{n}}\max_{t\in T}\|e(t)\|$. Let $q_i = 1$ if $\sum_{t\in T} z_i(t)\mu(t) > 0$ and $q_i = \frac{1}{\sqrt{n}}$ otherwise.

Let $K_1 = \{k \in (1,\ldots,\ell): z_k > 0\}$ and $K_2 = \{k \in (1,\ldots,\ell): z_k \le 0\}$. Then

$$q \cdot \sum_{t\in T} z(t)\mu(t) = \sum_{i\in K_1} [\sum_{t\in T} z_i(t)\mu(t)] + \frac{1}{\sqrt{n}} \sum_{i\in K_2} [\sum_{t\in T} z_i(t)\mu(t)]$$

$$\le \frac{1}{\sqrt{n}} \max_{t\in T} \|e(t)\|.$$

But,

$$\sum_{i\in K_1} [\sum_{t\in T} z_i(t)\mu(t)] = \sum_{i=1}^{\ell} \max\{\sum_{t\in T} z_i(t)\mu(t), 0\}$$

$$\le \frac{\ell}{\sqrt{n}} \max_{t\in T} \|e(t)\| - \frac{1}{\sqrt{n}} \sum_{i\in K_2} [\sum_{t\in T} z_i(t)\mu(t)]$$

$$\le \frac{\ell}{\sqrt{m}} \max_{t\in T} \|e(t)\| + \frac{1}{\sqrt{n}} \sum_{i=1}^{\ell} [|\sum_{t\in T} z_i(t)\mu(t)|]$$

$$= \frac{\ell}{\sqrt{n}} \max_{t\in T} \|e(t)\| + \frac{1}{\sqrt{n}} \|z\|$$

$$\le \frac{\ell}{\sqrt{n}} \max_{t\in T} \|e(t)\| + \frac{1}{\sqrt{n}} \max_{t\in T} \|e(t)\|$$

$$= \frac{1}{\sqrt{n}} (\ell + 1) \max_{t\in T} \|e(t)\|.$$

Thus, $\sum_{i=1}^{\ell} \max\{\sum_{t\in T} z_i(t)\mu(t), 0\} \le \frac{1}{\sqrt{n}} (\ell + 1) \max_{t\in T} \|e(t)\|$ and since $z(t_j)$ is chosen arbitrarily to be in $D(p,t_j)$, then $z(t) \in D(p,t)$ for all $t \in T$. This completes the proof.■

Following Anderson's (1978) arguments one can easily extend his result to small and large agents. More formally we have:

PROPOSITION 6.2. *Let* $\bar{\mathcal{E}}: T \to \mathcal{P} \times R_+^{\ell}$ *be a finite mixed exchange economy. Let* $\succ_t \in \mathcal{P}$ *satisfy* (iv) *and* (v) *for all* $t \in T$ *and* (vi) *for all* $t \in T_1$. *If* $f \in C(\bar{\mathcal{E}})$, *then there exists* $p \in P$ *such that* (i) $\sum_{t\in T} |p\cdot(f(t)\mu(t) - e(t)\mu(t))| \le \frac{2M}{n}$, *and* (ii) $\sum_{t\in T} |\inf\{p\cdot x(t)\mu(t) - p\cdot e(t)\mu(t): x(t) \succ_t f(t)\}| \le \frac{2M}{n}$, *where* $M = \sup\{\|e(t_1) + \ldots + e(t_\ell)\|_\infty : t_i \in T_0, i = 1,\ldots,\ell\}$.

PROOF. Define $\phi(t) = \{x - e(t) : x \succ_t f(t)\} \cup \{0\}$ for all $t \in T$ and $\Phi = \sum_{t \in T} \phi(t)\mu(t)$. Let $z = \frac{Me}{n}$; we need to show that con $\Phi \cap \{w \in R^{\ell} : w \ll -z\} = \emptyset$. Suppose that there is a $G \in$ con Φ, $G \ll -z$. Then there exists $g : T \to R^{\ell}$ with $g(t) \in$ con $\phi(t)$ for all $t \in T$, such that $G = \sum_{t \in T} g(t)\mu(t)$. Since for all $t \in T_1$ $\succ_t$ is convex, con $\Phi = \sum_{t \in T_0}$ con $\phi(t)\mu(t) + \sum_{t \in T_1} \phi(t)\mu(t)$. By the Shapley-Folkman theorem we have that $g(t) \in$ con $\phi(t)$ for all $t \in T_0$, $g(t) \in \phi(t)$ for all $t \in T_1$ and $|T_0'| = |\{t \in T_0 : g(t) \notin \phi(t)\}| \le \ell$. Define $y = \sum_{t \in T \setminus T_0'} g(t)\mu(t)$. Since $0 \in \phi(t)$, we have that $y \in \Phi$. Since $\phi(t) \ge -e(t)$, then con $\phi(t) \ge -e(t)$. Hence, $y = G - \sum_{t \in T_0'} g(t)\mu(t) \le G + \sum_{t \in T_0'} e(t)\mu(t) \le G + \frac{1}{n} \sum_{i=1}^{\ell} e(t_i)$ $\le G + \frac{Me}{n} = G + z \ll 0$. Let $V = \{t \in T : g(t) \ne 0\}$. Define $h(t) = g(t) + e(t) - \frac{G}{\mu(V)}$ for all $t \in V$. Then $h(t) \gg g(t) + e(t)$ and $g(t) + e(t) \succ_t f(t)$ for all $t \in V$. By (iv) $h(t) \succ_t f(t)$ for all $t \in V$. Moreover, it can be easily checked that $\sum_{t \in V} h(t)\mu(t) = \sum_{t \in V} e(t)\mu(t)$. Hence, $h(t)$ blocks $f(t)$ via V. This contradiction establishes that if $G \ll -z$, $G \notin$ con Φ. By the separating hyperplane theorem there exists a price system $p \in P$ such that p separates Φ from $\{w \in R^{\ell} : w \ll -z\}$. Hence, $\inf p \cdot \Phi \ge \sup\{p \cdot w : w \ll -z\} = -p \cdot z = -\frac{M}{n}$. Since $0 \in \phi(t)$ for all $t \in T$, $0 \ge \sum_{t \in T} \inf p \cdot \phi(t)\mu(t) = \inf p \cdot \Phi \ge -\frac{M}{n}$. By (v) for every $\varepsilon > 0$, $f(t) - e(t) + \varepsilon \in \phi(t)$ and $p \cdot (f(t) - e(t) + \varepsilon) \ge \inf p \cdot \phi(t)$. Let $B = \{t \in T : p \cdot f(t) < p \cdot e(t)\}$, then $\sum_{t \in B} p \cdot [f(t)\mu(t) - e(t)\mu(t)] + \varepsilon \ge \sum_{t \in B} \inf p \cdot \phi(t)\mu(t) \ge -\frac{M}{n}$; and by letting ε converge to zero, we have that $\sum_{t \in B} p \cdot (f(t)\mu(t) - e(t)\mu(t)) \ge \sum_{t \in B} \inf p \cdot \phi(t)\mu(t) \ge -\frac{M}{n}$. Clearly, $\sum_{t \in T} p \cdot (f(t)\mu(t) - e(t)\mu(t)) = p \cdot \sum_{t \in T} (f(t)\mu(t) - e(t)\mu(t)) = p \cdot 0$. Hence

$$\sum_{t \in T} |p \cdot (f(t)\mu(t) - e(t)\mu(t))| = 2 \sum_{t \in B} |p \cdot (f(t)\mu(t) - e(t)\mu(t))| \le \frac{2M}{n}.$$

Furthermore,

$$\sum_{t \in T} |\inf\{p \cdot x(t)\mu(t) - p \cdot e(t)\mu(t) : x(t) \succ_t f(t)\}| \le -\sum_{t \in T} \inf p \cdot \phi(t)\mu(t) + \sum_{t \notin B} p \cdot (f(t)\mu(t) - e(t)\mu(t)) \le \frac{2M}{n}.$$

This completes the proof of Proposition 6.2. ∎

COROLLARY 6.1. *The conclusion of Proposition 6.2 remains true if* $T_1 = \emptyset$.

Let $\mathcal{E}$ be a finite exchange economy where $T_1 = \emptyset$, i.e., $\mathcal{E}: T_0 \to \mathcal{P} \times R_+^\ell$.

Denote by $C_0(\mathcal{E})$ the set of all core allocations in $\mathcal{E}$ and by $N_0(\mathcal{E})$ the set of all nondiscriminatory allocations in $\mathcal{E}$.

PROOF OF THEOREM 6.2. Combine Proposition 2.8 and Corollary 6.1.∎

PROOF OF THEOREM 6.3. Let f be a function of T into R_+^ℓ and $p \in \Delta$ a price vector such that $f(t)$ is maximal for $\succ_t$ in the budget set for all $t \in T$ and $\sum_{i=1}^{\ell} \max\{\sum_{t\in T} f_i(t)\mu(t) - \sum_{t\in T} e_i(t)\mu(t), 0\} \leq \frac{1}{\sqrt{n}}(\ell+1)\max_{t\in T}\|e(t)\|$, i.e., (p,f) constitutes a near or approximate competitive equilibrium for $\bar{\mathcal{E}}$. Denote by $NE(\bar{\mathcal{E}})$ the set of near competitive equilibrium allocations in $\bar{\mathcal{E}}$. Also, denote by $AN(\bar{\mathcal{E}})$ the set of approximate nondiscriminatory allocations in $\bar{\mathcal{E}}$. Then repeating the argument used in the proof of Theorem 6.1, we have that $NE(\bar{\mathcal{E}}) \subseteq AN(\bar{\mathcal{E}})$. But since by Proposition 6.1 $NE(\bar{\mathcal{E}}) \neq \emptyset$ then we have $AN(\bar{\mathcal{E}}) \neq \emptyset$, and this completes the proof.∎

7. PERFECTLY COMPETITIVE ECONOMIES

Perfect competition prevails in an economy if and only if no agent is able to affect the prices at which other agents buy and sell goods. However, if the set of agents is finite, then each trader in the economy may have a significant effect on aggregate demand or supply and can affect prices. Consequently, the notion of perfect competition may break down. To resolve this problem Aumann (1964) introduced the continuum of agents, i.e., the set of agents is described by an atomless measure space. In this case each agent in the economy is negligible, i.e., has measure zero (recall then an atom does not "split" into smaller non-null pieces) and a priori takes prices as given. Thus, an economy with an atomless measure space of agents captures the meaning of perfect competition. It is the purpose of this section to examine perfectly competitive economies, i.e., economies with an atomless measure space of economic agents. In particular using some technical results of Loeb (1975) and Anderson (1976, 1982) we will show how the theorems for economies with a finite number of agents of the previous section, can be extended to economies with an atomless measure space of agents.

7.1. *Definitions and Results*

For a more complete treatment of several notions used in this section we refer the reader to Emmons (1984), Emmons-Yannelis (1984), and Yannelis (1983). For an introduction to Nonstandard Analysis and Loeb measure spaces see Loeb (1979).

A *Loeb exchange economy* is a $L(\tau)$-measurable mapping $\mathcal{E}_L:(T,L(\tau),L(\mu)) \to \mathcal{P} \times R^{\ell}_{+}$, where $(T,L(\tau),L(\mu))$ is a *-finite Loeb measure space of agents (Loeb measure spaces are general enough to contain atoms and an atomless part). Let $\succ_t$ be the projection of $\mathcal{E}_L(t)$ onto $\mathcal{P}$ and $e(t)$ be the projection of $\mathcal{E}_L(t)$ onto R^{ℓ}_{+}. Clearly $\succ_t$ denotes the preference of agent t and $e(t)$ his/her initial endowment. An *assignment* x is an $L(\mu)$-integrable function of T into R^{ℓ}_{+}. An assignment $f:T \to R^{\ell}_{+}$ and a price vector $p \in \text{int } R^{\ell}_{+}$ constitute a *competitive equilibrium* for $\mathcal{E}_L$ if

(i) $f(t)$ is maximal for $\succ_t$ in $\{x \in R^{\ell}_{+}: p\cdot x \leq p\cdot e(t)\}$, $L(\mu)$ - a.e.

(ii) $\int_T f dL(\mu) = \int_T e dL(\mu)$.

An *allocation* is an assignment $x:T \to R^{\ell}_{+}$ such that $\int_T x dL(\mu) = \int_T e dL(\mu)$. An allocation $f:T \to R^{\ell}_{+}$ is said to be *nondiscriminatory* for the economy $\mathcal{E}_L$ if there exists no assignment $y:T \to R^{\ell}_{+}$ and disjoint coalitions S_1, S_2 such that $y(t) \succ_t f(t)$ $L(\mu)$-almost all t in S_1 and $\int_{S_1}(y(t) - e(t))dL(\mu) = \int_{S_2}(f(t) - e(t))dL(\mu)$. Denote by $N(\mathcal{E}_L)$ the set of all nondiscriminatory allocations in $\mathcal{E}_L$.

THEOREM 7.1. *Let* $\mathcal{E}_L:(T,L(\tau),L(\mu)) \to \mathcal{P} \times R^{\ell}_{+}$ *be a Loeb exchange economy satisfying the following conditions:*

(1) $(T,L(\tau),L(\mu))$ *is a *-finite atomless Loeb measure space*

(2) $\succ_t \in \mathcal{P}$ *is irreflexive, transitive, monotone, continuous*

(3) $\succ_t \in \mathcal{P}'$, *where* $\mathcal{P}'$ *is a compact subset of* $\mathcal{P}$ *in the topology of closed convergence (see Hildenbrand, (1974)).*

(4) $\int_T e(t)dL(\mu) \gg 0$.

Then if $f \in N(\mathcal{E}_L)$, *there exists* $p \in R^{\ell}_{+}\backslash\{0\}$ *such that (i) and (ii) are satisfied.*

PROOF. By Anderson's (1982, Theorem 5.3) "lifting" theorem there exists an internal map $*\mathcal{E}:T \to *\mathcal{P} \times *R_+^\ell$ such that $°(*\mathcal{E}(t)) = °(*\succ_t, *e(t)) = \mathcal{E}_L(t) = (\succ_t, e(t)), L(\mu)$ - a.e. (where ° denotes standard part). In other words the standard part of the internal nonstandard exchange economy $*\mathcal{E}$ is the same with the Loeb exchange economy $\mathcal{E}_L$.

By the transfer of Theorem 6.2 we have that if $f \in N(*\mathcal{E})$, then there exists $p \in *R_+^\ell \setminus \{0\}$ such that

$$\frac{1}{|T|}\sum_{t\in T}|p\cdot(f(t) - *e(t))| \le \frac{2M}{|T|} \simeq 0, \text{ where } |T| = \omega \in *N - N \quad (7.1)$$

and

$$\begin{aligned}&\frac{1}{|T|}\sum_{t\in T}|\inf\{p\cdot(x - e'(t)) : x \overset{*}{\succ}_t f(t)\}| \le \frac{2M}{|T|} \simeq 0, \text{ i.e.,}\\ &p\cdot f(t) \simeq p\cdot *e(t) \simeq \inf\{p\cdot x : x \overset{*}{\succ}_t f(t)\} \text{ for all } t \in K \text{ where}\\ &K \text{ is an internal set of traders such that } \frac{|K|}{\omega} \simeq 1.\end{aligned} \quad (7.2)$$

The economy $*\mathcal{E}$ will satisfy the following assumptions:

*(1) $\sum_{t\in T}{}^{*}e(t) \gg 0$

*(2) $*\mathcal{P}'$ is near standard, i.e., preferences lie in the nonstandard extension of $\mathcal{P}'$ which is compact in the topology of closed convergence.

It follows from assumption *(1) and the continuity of preferences that (7.2) can be strengthened so $f(t)$ is maximal for $\overset{*}{\succ}_t$ in $\{x : p\cdot x \le p\cdot *e(t)\}$ for all $t \in K$ where K is an internal set of agents such that $|K|/\omega \simeq 1$. By a standard argument one can show that $p \gg 0$ (see Khan (1975) for a complete proof). It can be easily shown that $f:T \to *R_+^\ell$ is S-integrable as well.

Notice, that for any internal set S, $\frac{|S|}{\omega} \simeq 0$ we have that $p\cdot\frac{1}{\omega}\sum_{t\in S}f(t) = \frac{1}{\omega}\sum_{t\in S}p\cdot f(t) \simeq \frac{1}{\omega}\sum_{t\in S}p\cdot *e(t) \simeq 0$, since $*e:T \to *R_+^\ell$ is S-integrable. Therefore, since $p \gg 0$, it follows that $f:T \to *R_+^\ell$ is S-integrable. We will now need the following two Lemmas.

LEMMA 7.1. *The pair* (f,p) *constitutes a competitive equilibrium for* $*\mathcal{E}$, *if and only if* $(°p, °f)$ *is a competitive equilibrium for* $\mathcal{E}_L$.

PROOF. (⇒) Since the pair (p,f) is a competitive equilibrium for $*\mathcal{E}$ we have that:

$$\frac{1}{\omega}\sum_{t\in T} f(t) \simeq \frac{1}{\omega}\sum_{t\in T} {}^*e(t), p\cdot f(t) \le p\cdot {}^*e(t) \text{ and } y \mathrel{{}^*\!\!\succ_t} f(t) \Rightarrow p\cdot y >$$

$p\cdot {}^*e(t)$ for all $t \in K$ where K is an internal set of agents, $|K|/\omega \simeq 1$.

Suppose that $({}^\circ p, {}^\circ f)$ is not a competitive equilibrium for $\mathcal{E}_L$. Then there exists $g: T \to R_+^\ell$ $L(\mu)$-integrable such that

$$\int_T g\,dL(\mu) = \int_T e\,dL(\mu) \quad \text{and} \tag{7.3}$$

$${}^\circ p\cdot g \le {}^\circ p \cdot e(t) \text{ and } g \succ_t {}^\circ f(t) \text{ for all } t \in S,\ L(\mu)(S) > 0. \tag{7.4}$$

By Theorem 7 in Anderson (1976) there exists a function $x: T \to {}^*R_+^\ell$ such that x is S-integrable and ${}^\circ x = g$, $L(\mu)$ – a.e. From (7.4) it follows that

$${}^\circ p\cdot {}^\circ x \le {}^\circ p \cdot e(t) \text{ and } {}^\circ x \succ_t {}^\circ f(t) \text{ for all } t \in S,\ L(\mu)(S) > 0.$$

Since ${}^\circ x$ is $L(\mu)$-integrable and ${}^\circ x = g$, $L(\mu)$ – a.e., $\int_T x\,dL(\mu) = \int_T g\,dL(\mu) = \int_T e\,dL(\mu)$. Again by Theorem 7 in Anderson (1976) we have that $\int_T {}^\circ x\,dL(\mu) \simeq \frac{1}{\omega}\sum_{t\in T} x(t)$. Hence, since $\int_T e\,dL(\mu) \simeq \frac{1}{\omega}\sum_{t\in T} {}^*e(t)$ by (7.3) we have that $\frac{1}{\omega}\sum_{t\in T} x(t) \simeq \frac{1}{\omega}\sum_{t\in T} {}^*e(t)$; and from (7.4) it follows that $p\cdot x \le p\cdot {}^*e(t)$ and $x \mathrel{{}^*\!\!\succ_t} f(t)$ for all $t \in S$, $\frac{|S|}{\omega} \not\simeq 0$, a contradiction to the fact that (p,f) is a competitive equilibrium for $*\mathcal{E}$.

(⇐) Let $({}^\circ p, {}^\circ f)$ be a competitive equilibrium for $\mathcal{E}_L$ and the pair (p,f) is not a competitive equilibrium for $*\mathcal{E}$. Then there exists an internal function $y: T \to {}^*R_+^\ell$ such that

$$p\cdot y \le p\cdot {}^*e(t) \text{ and } y \mathrel{{}^*\!\!\succ_t} f(t) \text{ for all } t \in S,\ S \text{ internal},\ \frac{|S|}{\omega} \not\simeq 0. \tag{7.5}$$

Since $p\cdot y \le p\cdot {}^*e(t)$ for all t in the noninfinitesimal internal set S, it follows that for any internal V, $\frac{|V|}{\omega} \simeq 0$ $p\cdot \frac{1}{\omega}\sum_{t\in V} y(t) =$

$\frac{1}{\omega}\sum_{t\in V} p\cdot y(t) \leq \frac{1}{\omega}\sum_{t\in V} p\cdot {}^*e(t) \simeq 0$ since e is S-integrable. Since $p \gg\!\!\!\!/ \; 0$ it follows that $\frac{1}{\omega}\sum_{t\in V} y(t) \simeq 0$. Hence, $y:T \to {}^*R^{\ell}_{+}$ is S- integrable and by Theorem 6 in Anderson (1976) $°y$ is $L(\mu)$-integrable. It follows from (7.5) that $°p\cdot°y(t) \leq °p\cdot e(t)$ and $°y(t) \succ_t °f(t)$ for all $t \in S$, $L(\mu)(S) > 0$; a contradiction to the fact that $(°p,°f)$ is a competitive equilibria for $\mathcal{E}_L$. This completes the proof of the Lemma.∎

LEMMA 7.2. $f \in N({}^*\mathcal{E}) \Leftrightarrow °f \in N(\mathcal{E}_L)$.

PROOF. ($\Leftarrow$) Let $°f \in N(\mathcal{E}_L)$ and $f \notin N({}^*\mathcal{E})$. Then there exist an internal function $y:T \to {}^*R^{\ell}_{+}$ and disjoint coalitions S_1, S_2 such that

$$y(t) \succsim_t f(t) \quad \text{for all} \quad t \in S_1, \quad \text{and} \tag{7.6}$$

$$\frac{1}{\omega}\sum_{t\in S_1}(y(t) - {}^*e(t)) \simeq \frac{1}{\omega}\sum_{t\in S_2}(f(t) - {}^*e(t)) \tag{7.7}$$

It can be easily seen that $y:T \to {}^*R^{\ell}_{+}$ is S-integrable. In fact, since ${}^*e:T \to {}^*R^{\ell}_{+}$ and $f:T \to {}^*R^{\ell}_{+}$ are S-integrable it follows directly from (7.7) that for any internal set V, $\frac{|V|}{\omega} \simeq 0$, $\frac{1}{\omega}\sum_{t\in V} y(t) \simeq 0$. By Theorem 6 in Anderson (1976) $°y$ is $L(\mu)$-integrable and $°(\frac{1}{\omega}\sum y(t)) = \int °y\,dL(\mu)$. Since $\int °y\,dL(\mu) \simeq \frac{1}{\omega}\sum y(t)$ and $\int e\,dL(\mu) \simeq \frac{1}{\omega}\sum {}^*e(t)$ it follows from (7.7) that $\int_{S_1}(°y(t) - e(t))dL(\mu) = \int_{S_2}(°f(t) - e(t))dL(\mu)$. Moreover, from (7.6) it follows that $°y(t) \succ_t °f(t)$ for all $t \in S_1$. But this contradicts the fact that $°f \in N(\mathcal{E}_L)$.

($\Rightarrow$) Let $f \in N({}^*\mathcal{E})$ and $°f \notin N(\mathcal{E}_L)$. Then there exist $y:T \to R^{\ell}_{+}$ $L(\mu)$-integrable and S_1, S_1, $S_1 \cap S_2 = \emptyset$ such that

$$y(t) \succ_t °f(t) \quad \text{for all} \quad t \in S_1, \quad \text{and} \tag{7.8}$$

$$\int_{S_1}(y(t) - e(t))dL(\mu) = \int_{S_2}(°f(t) - e(t))dL(\mu). \tag{7.9}$$

Since y is $L(\mu)$-integrable there exists $x:T \to {}^*R^{\ell}_{+}$ S-integrable such that $°x = y$, $L(\mu)$ - a.e. Since $°x$ is $L(\mu)$-integrable then

$\int {}^\circ x dL(\mu) = \int y dL(\mu)$, and it follows from (7.9) that

$$\int_{S_1} ({}^\circ x(t) - e(t)) dL(\mu) = \int_{S_2} ({}^\circ f(t) - e(t)) dL(\mu). \qquad (7.10)$$

From (7.8) it follows that $x(t) \ {}^*\!\succ_t f(t)$ for all $t \in S_1$. Furthermore, since $\int {}^\circ x dL(\mu) \simeq \frac{1}{\omega} \sum x(t)$, $\int {}^\circ f dL(\mu) \simeq \frac{1}{\omega} \sum f(t)$ and $\int e dL(\mu) \simeq \frac{1}{\omega} \sum {}^* e(t)$ it follows from (7.10) that $\frac{1}{\omega} \sum_{t \in S_1} (x(t) - {}^*e(t)) \simeq \frac{1}{\omega} \sum_{t \in S_2} (f(t) - {}^*e(t))$ a contradiction to the fact that $f \in N({}^*\mathcal{E})$.■

It follows from Lemmas 7.1 and 7.2 that the proof of Theorem 7.1 is now complete.

A couple of methodological comments are in order. The purpose of Theorem 7.1 (which is indeed an equivalence result, recall that all competitive equilibrium allocations are nondiscriminatory) was not to prove a new result. As a matter of fact Theorem 7.1 is proved in Gabszewicz (1975) through the core equivalence result of Aumann (1964). Our purpose was to demonstrate the importance and powerfulness of the Loeb-Anderson methodology. Specifically this technique allows us to extend results from finite economies into economies with infinitely many agents in a rather simple and intuitive way. Since some results may be simpler to prove in a finite economy framework the Loeb-Anderson methodology seems to be very promising in obtaining results for perfectly competitive economies "very cheaply." It should be noted that Theorem 7.1 involves a restriction on the measure space of agents, i.e., it is an atomless Loeb measure space and not any arbitrary atomless measure space as it has been the case in the literature (see for instance Richter (1971) or Armstrong-Richter (1984) for a complete discussion on the space of agents). However, if the purpose of the nonatomicity of the measure space is to capture the meaning of perfect competition then, atomless Loeb measure spaces serve precisely our purpose.

Finally, we would like to point out that the results of Section 6 can be extended accordingly to economies with a continuum of agents using the above technique (see Yannelis (1983b) for complete proofs).

REFERENCES

Anderson, R. M. (1982), "Star-Finite Representation of Measure Spaces," *Transactions of the American Mathematical Society*, 271, 667-687.

Anderson, R. M. (1978), "An Elementary Core Equivalence Theorem," *Econometrica*, 46, 1483-1487.

Anderson, R. M. (1976), "A Non-Standard Representation for Browian Motion and Itô Integration," *Israel Journal of Mathematics*, 25, 15-46.

Anderson, R. M., M. Ali Khan and S. Rashid (1983), "Approximate Equilibria with Bounds Independent of Preferences," *Review of Economic Studies*, XLIV, 473-475.

Armstrong, T. E. and M. K. Richter (1984), "The Core-Walras Equivalence," *Journal of Economic Theory*, 33, 116-151.

Aumann, R. J. (1982), "On the Non-Transferable Utility Value: A Comment on the Roth-Shafer Examples," *Econometrica*, (forthcoming).

Aumann, R. J. (1975), "Values of Markets with a Continuum of Traders," *Econometrica*, 43, 611-646.

Aumann, R. J. (1964), "Markets with a Continuum of Traders," *Econometrica*, 32, 39-50.

Bennet, E. (1983), "The Aspiration Approach to Predicting Coalition Formation and Payoff Distribution in Scale Payment Games," *International Journal of Game Theory*, 12, 1-28.

Bennet, E. and W. R. Zame (1983), "Bargaining Aspirations for NTU Games," (mimeograph), SUNY-Buffalo.

Brown, D. J. and L. Lewis (1981), "Existence of Equilibrium in a Hyperfinite Exchange Economy," C.F.D.P. #581.

Champsaur, P. (1975), "Cooperation vs. Competition," *Journal of Economic Theory*, 11, 394-417.

Cheng, H. (1981), "On Dual Regularity and Value Convergence Theorems," *Journal of Mathematical Economics*, 8, 37-57.

Debreu, G. and H. Scarf (1963), "A Limit Theorem on the Core of an Economy," *International Economic Review*, 4, 236-246.

Emmons, D. (1984), "Existence of Lindahl Equilibria, without Ordered Preferences," *Journal of Economic Theory*, 34, 342-359.

Emmons, D. and A. J. Scafuri (1984), "Value Allocations - An Exposition." In: *Advances in Equilibrium Theory* (Edited by C. D. Aliprantis, O. Burkinshaw and N. J. Rothman), Springer-Verlag Lecture Notes in Economics and Mathematical Systems, 1985.

Emmons, D. and N. C. Yannelis (1984), "On Perfectly Competitive Economies: Loeb Economies." In: *Advances in Equilibrium Theory* (Edited by C. D. Aliprantis, O. Burkinshaw and N. J. Rothman), Springer-Verlag Lecture Notes in Economics and Mathematical Systems, 1985.

Feldman, A. and A. Kirman (1974), "Fairness and Envy," *American Economic Review*, 64, 995-1005.

Gabszewicz, J. J. (1975), "Coalitional Fairness of Allocations in Pure Exchange Economies," *Econometrica*, 43, 661-668.

Harsanyi, J. C. (1980), "Comments on Roth's Paper: 'Values for Games Without Side-Payments,'" *Econometrica*, 48, 477.

Hildenbrand, W. (1974), *Core and Equilibria of a Large Economy*, Princeton University Press, 1974.

Hurwicz, L. (1972), "On Informationally Decentralized Systems," in C. B. McGuire and R. Radner, (editors), *Decision and Organization*, North Holland, Amsterdam, 297-336.

Khan, M. Ali (1975), "Some Approximate Equilibria," *Journal of Mathematical Economics*, 2, 63-86.

Loeb, P. (1979), "An Introduction to Non-Standard Analysis and Hyperfinite Probability Theory," *Probabilistic Analysis and Related Topics*, Vol. 2, A. T. Bharucha-Reid, (editor), Academic Press, New York, 105-142.

Loeb, P. (1975), "Conversion from Non-Standard to Standard Measure Spaces and Applications to Probability Theory," *Transactions of the American Mathematical Society*, 211, 113-122.

Mas-Colell, A. (1977), "Competitive and Value Allocations of Large Exchange Economies," *Journal of Economic Theory*, 14, 419-438.

Rashid, S. (1979), "The Relationship Between Measure-Theoretic and Non-Standard Exchange Economies," *Journal of Mathematical Economics*, 5, 195-202.

Richter, M. K. (1971), "Coalitions, Cores and Competition," *Journal of Economic Theory*, 3, 323-334.

Roth, A. (1983), "Some Further Thoughts on the NTU Value and Related Matters: A Rejoinder to Aumann," (mimeograph), University of Pittsburg.

Roth, A. (1980), "Values for Games Without Side Payments: Some Difficulties with Current Concepts," *Econometrica*, 48, 457-465.

Scafuri, A. J. and N. C. Yannelis (1984), "Non-Symmetric Cardinal Value Allocations," *Econometrica*, 52, 1365-1368.

Shafer, W. J. (1980), "On the Existence and Interpretation of Value Allocation," *Econometrica*, 48, 467-474.

Shapley, L. A. (1969), "Utility Comparisons and the Theory of Games," in La Decision, edited by Guilbaud, Editions du CNRS, Paris, France.

Thomson, W. (1983), "The Manipulability of Shapley Value," *International Journal of Game Theory*, (forthcoming).

Thomson, W. (1979), "The Equilibrium Allocations of Walras and Lindahl Manipulation Games," (mimeograph), University of Minnesota.

Thomson, W. and H. Varian (1984), "Theories of Justice Based on Symmetry." In: *Social Goals and Social Organization* (Edited by L. Hurwicz, D. Schmeidler and H. Sonnenschein), Cambridge University Press, 1985.

Varian, H. (1974), "Equity, Envy and Efficiency," *Journal of Economic Theory*, 9, 63-91.

Yannelis, N. C. (1983a), "Existence and Fairness of Value Allocation Without Convex Preferences," *Journal of Economic Theory*, 31, 282-292.

Yannelis, N. C. (1983b), "Solution Concepts in Economic Theory," Doctoral Dissertation, The University of Rochester.

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