Non-emptiness of the alpha-core*

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Abstract

We prove nonemptiness of the α-core for balanced games with nonordered preferences, extending and generalizing in several aspects the results of Scarf (1971), Border (1984), Florenzano (1989), Yannelis (1991b) and Kajii (1992). In particular we answer an open question in Kajii (1992) regarding the applicability of the nonemptiness results to models with infinite dimensional strategy spaces. We provide two models with Knightian and voting preferences for which the results of Scarf (1971) and Kajii (1992) cannot be applied while our nonemptiness result applies.

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1 Introduction

The core is one of the most prominent solution concepts in cooperative game theory. It applies to settings in which players act cooperatively within each coalition but compete

*The first draft of this paper dates back to 2010, when Filipe was visiting the Economics Department at the University of Illinois at Urbana-Champaign. The project was set aside for many years, until renewed interest motivated a fresh revision. We are especially thankful to Rabah Amir for drawing our attention to the special issue in honor of Myrna Wooders—an occasion that inspired us to contribute to her well-deserved festschrift. This paper is dedicated to Myrna in recognition of her pioneering contributions to the theory of the core and her steadfast dedication to economic theory over several decades. Rabah Amir's encouragement was instrumental in bringing this long-standing project to completion.

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across coalitions. A feasible allocation for the grand coalition, also called a social state, belongs to the core if no sub-coalition has an incentive to form and deviate from it.

In economic environments with externalities, an agent's payoff depends not only on her own action but also on the actions of others. In such settings, defining deviations (or *blocking*) becomes subtle, as the reactions of outsiders influence the effectiveness of a coalition's deviation. Indeed, when a coalition plans to block a given social state, the actions of agents outside the coalition affect the welfare of its members. Thus, coalition members must anticipate how outsiders might respond.

One classical approach assumes that outsiders maintain their current strategies while the coalition attempts to improve its welfare. This corresponds to the concept of a strong Cournot–Nash equilibrium, which often fails to guarantee existence because coalitions can block too easily. Instead, Aumann (1961) proposed a more conservative solution concept in which coalition members expect that outsiders may adapt or retaliate. Formally, a coalition is compelled to change its strategy only if, for every possible reaction of the counter-coalition, each member strictly prefers the resulting outcome to the current social state. If this condition holds, the coalition is said to α_{K}-block the social state.[1] A social state belongs to the α_{K}-core if no coalition can α_{K}-block it. Under this concept, each agent in a blocking coalition assumes that outsiders have considerable flexibility in their response. As a result, α_{K}-blocking becomes more difficult, and the α_{K}-core tends to be large. Scarf (1971) established the nonemptiness of the α_{K}-core for normal-form games in which agents' preferences are represented by continuous and quasi-concave utility or payoff functions.

Border (1984) showed that completeness and transitivity of preferences are inessential for core nonemptiness in games without preference externalities. Building on this perspective, Kajii (1992) examined whether these properties are necessary for the validity of Scarf (1971)'s theorem. He proved that games with possibly nonordered preferences also have a nonempty α_{K}-core, provided that preferences are continuous and the set of feasible strategies is compact in a topology derived from a norm. This additional requirement is innocuous in finite-dimensional strategy spaces.[2] However, as Kajii (1992) himself noted, the norm-

[1] The subscript K refers to the formulation introduced by Kajii (1992), who extended Aumann's core concept to games with coalition-specific feasibility constraints, also known as *generalized games*.

[2] Any Hausdorff linear topology on a finite-dimensional vector space coincides with the Euclidean topology.

compactness assumption on the feasible strategy sets significantly limits the applicability of his nonemptiness result in games with infinite-dimensional strategy spaces. Mas-Colell and Zame (1991) offer a detailed discussion of this issue; in particular, they show that for games derived from exchange economies, the set of feasible trades is generally only weakly compact, not norm-compact.[3]

It is also worth noting that the nonemptiness result in Kajii (1992) does not subsume the existing results in the literature as special cases. Specifically, Florenzano (1989) proved that if preference relations are nonordered but exhibit weak externalities in consumption, then for games derived from exchange economies, the α_{K}-core is non-empty under compactness and continuity assumptions for any Hausdorff linear topology. The same level of generality with respect to the topology is achieved by Scarf (1971) for games in which preference relations may include externalities but are ordered (i.e., representable by utility or payoff functions).[4] Kajii (1992) investigates games that are more general than those in the aforementioned studies: preference relations are nonordered (as in Florenzano (1989)) and include consumption externalities (as in Scarf (1971)). Nevertheless, the existence result in Kajii (1992) does not encompass the findings of either Scarf (1971) or Florenzano (1989), since it relies on the additional assumption that the topology is normable.

One might think that the additional (and restrictive) assumption imposed by Kajii (1992) is the "price to pay" for combining nonordered preferences with externalities. One of the main contributions of this paper is to demonstrate that this is not the case. We prove that nonemptiness of the α_{K}-core can be established for any Hausdorff linear topology, including weak topologies that are crucial for ensuring compactness of the feasible strategy sets in infinite dimensions. Our theorem thus generalizes and unifies the results of Scarf (1971), Border (1984), Florenzano (1989), and Kajii (1992). We also provide two illustrative applications that showcase its economic relevance.

We further contribute by introducing a new solution concept that modifies the treatment of outsiders in the blocking test. Relative to Kajii (1992), outsiders are granted less freedom to react, which strengthens coalitional deviations and sharpens the comparison across cores.

[3] In Section 6, we provide two additional examples to illustrate this point.

[4] Strictly speaking, Scarf (1971) established nonemptiness only for finite-dimensional strategy spaces. However, his arguments can be straightforwardly extended to any Hausdorff topological vector space. See Section 4.2 for a detailed discussion.

In particular, we prove the nonemptiness of a core that is strictly smaller than the α_{K}-core. Several alternative solution concepts to that of Aumann (1961) and Kajii (1992) have been proposed. We previously mentioned the strong Cournot–Nash equilibrium, in which agents forming a blocking coalition assume that outsiders will not react. Another well-known concept is the β-core, where a blocking coalition is no longer required to choose its strategy independently of the remaining players but can instead condition its strategy on the behavior of the complementary coalition. Because the blocking power is greater, the β-core is smaller than the α_{K}-core. However, as shown by Scarf (1971), the β-blocking power is so strong that examples of empty β-cores are easy to construct.

A further refinement was introduced by Yannelis (1991b) in the context of games derived from exchange economies. In this framework, agents forming a blocking coalition must choose a joint strategy that is feasible for the coalition. There is no a priori justification for assuming that the counter-coalition is not similarly constrained to select feasible joint strategies. Following this reasoning, Yannelis (1991b) proposed an alternative definition of the α-core: a coalition is said to α_{Y}-block a social state if there exists a feasible joint strategy such that each coalition member prefers it to the current social state, regardless of the counter-coalition's feasible responses. In the extension of Aumann's concept to games proposed by Kajii (1992), members of a blocking coalition do not require that outsiders choose feasible strategies. By assuming that outsiders are more constrained in their reactions, it becomes easier to α_{Y}-block a given strategy, making the α_{Y}-core smaller than the α_{K}-core.

Yannelis (1991b) (see also Koutsougeras and Yannelis (1993)) proved nonemptiness of the α_{Y}-core for pure exchange economies with at most two agents. For games involving more than two agents, the α_{Y}-blocking power becomes too strong. Indeed, Holly (1994) provided an example of a pure exchange economy with three agents—satisfying standard assumptions—for which the α_{Y}-core is empty.

We introduce an alternative definition of the α-core in which outsiders of a blocking coalition are granted less freedom to react than in the framework proposed by Kajii (1992), but more freedom than in the model developed by Yannelis (1991b). We prove the nonemptiness of our α-core under very general conditions, which in particular imply the nonemptiness of the *standard* α_{K}-core. Moreover, when there are at most two agents, our α-core coincides with the α_{Y}-core. Therefore, we obtain a general nonemptiness result that has the additional and interesting feature of unifying the nonemptiness results for the α_{K}-core and the α_{Y}-core. To highlight the distinction between the three concepts, we construct in Section 5 a concrete example where the α_{Y}-core is strictly contained in our α-core, which in turn is strictly contained in the α_{K}-core.

Since its first version was released in 2011, this paper has been cited in several significant studies that further explore the properties of the α-core. Uyanik (2015) investigated the α-core in discontinuous games, building on our foundational results. Yang (2017, 2018) extended Kajii's theorem to games with infinitely many players, drawing upon our findings. Yang and Yuan (2019) define the hybrid solution of games with nonordered preferences and prove its existence theorem in Hausdorff topological vector space. Basile and Scalzo (2020) and Scalzo (2022) established necessary and sufficient conditions for the nonemptiness of the α-core in games with nonordered and discontinuous preferences. Crettez et al. (2022) propose a new notion of coalitional equilibrium, the strong hybrid solution, and prove its existence whenever preferences are partially quasi-transferable. Additionally, Song and Guo (2022) examined the existence of α-core solutions in games with either finite or infinite player sets, explicitly acknowledging the relevance of our work. Recent papers on the α-core with nonordered preferences and/or a continuum of agents include Yang (2020), Yang and Zhang (2021), and Yang and Song (2022). These contributions, however, do not allow for coalition–specific feasibility sets F^S in finite–player nontransferable utility games, and thus do not address the differences in blocking power that arise when feasibility depends on the deviating coalition.

This paper also relates closely to the influential work of Myrna Wooders, whose pioneering contributions have profoundly shaped the field of cooperative game theory, especially in settings involving large economies, externalities, and club structures. In her seminal work on large replica games, Wooders (1983) introduced the notion of the ε-core and demonstrated its nonemptiness under general conditions. Wooders and Zame (1984) extended this analysis to formalize the existence of approximate cores in large games. In Shubik and Wooders (1983) she explored replica games and economies with externalities, highlighting how approximate cores could still yield robust solution concepts in complex environments. Later, Kovalenkov and Wooders (2003) developed core existence results for games and economies with clubs, offering important insights into cooperative behavior in settings with shared consumption and collective goods. More recently, Allouch and Wooders (2017) investigated the nonemptiness of approximate cores in large games, reinforcing the durability of these concepts under increasing complexity. Another major strand of her research involves the f-core, a concept introduced by Kaneko and Wooders (1986) to analyze games with a continuum of players and finite coalitions (see also Hammond et al. (1989), Kaneko and Wooders (1989), Kaneko and Wooders (1996)). This line of inquiry remains active; for instance, Konishi and Simeonov (2025) establish the nonemptiness of the f-core without comprehensiveness. Our approach–establishing core existence under minimal topological and preference assumptions–resonates strongly with Myrna Wooders' program, as we also seek general conditions under which core-like solutions remain viable. By relaxing norm-compactness and encompassing nonordered preferences, our result contributes to the robust core literature initiated by Myrna Wooders and extends its reach to a broader class of games.

The remainder of the paper is organized as follows. Section 2 defines generalized games, and core solution concepts are introduced in Section 3. Sufficient conditions for nonemptiness are presented in Section 4, and the detailed proof of our main theorem is given in Section 7. Section 5 provides a worked example that demonstrates the strict inclusions among the three core concepts studied in this paper. In Section 6, we illustrate the economic relevance of our general result through two applications. Standard definitions of continuity properties for correspondences, along with the proofs of technical results, are deferred to the appendix.

2 Games with Feasibility Constraints

We consider a game in normal form with a finite set I of agents and coalition-specific feasibility constraints. This class of games is commonly referred to as *generalized games*.[5] A subset $E \subseteq I$ represents a coalition and we fix a subset $\mathbb{I}$ of the set of all non-empty subsets of I that represents the family of admissible coalitions.[6] We assume that the grand coalition I and each individual coalition $\{i\}$ belong to $\mathbb{I}$. Each agent i chooses an individual strategy x_i in his strategy set X_i.

[5] Our environment is close to the social coalition equilibrium framework of Ichiishi (1981).

[6] Usually it is assumed that $\mathbb{I}$ coincides with $2^I \setminus \{\emptyset\}$ the set of non-empty subsets of I. We allow for the possibility that some coalitions cannot form.

Assumption 2.1. For each agent i, the strategy set X_i is a non-empty and convex subset of a Hausdorff topological vector space L.

An element $x = (x_i)_{i \in I}$ of $X := \prod_{i \in I} X_i$ is called a joint strategy (or an allocation) and may be thought as a social state. If E is a subset of I, then we denote by X^E the product set $\prod_{i \in E} X_i$.[7] When $E = I \setminus \{i\}$, we write X^{-i} rather than $X^{I \setminus \{i\}}$. Given a subset $S \subseteq I$, if y belongs to X^S and w belongs to $X^{I \setminus S}$ then $z = (y, w)$ denotes the allocation defined by $\pi^S(z) = y$ and $\pi^{I \setminus S}(z) = w$, where for any $E \subseteq I$, the mapping $\pi^E : X \to X^E$ is the natural projection defined by $\pi^E(x) = (x_i)_{i \in E}$ for each $x = (x_i)_{i \in I} \in X$.[8] For each i, we denote by $\mathbb{I}(i)$ the collection of all coalitions $E \in \mathbb{I}$ containing i.

For each coalition $S \in \mathbb{I}$, and given a social state $x \in X$, there is a set $F^S(x) \subseteq X^S$ which represents the set of feasible joint strategies for the coalition S. We assume the grand-coalition feasibility correspondence $x \mapsto F^I(x)$ is constant, and denote its value by F. We make the standard continuity, convexity and compactness assumption on the sets of feasible strategies.

Assumption 2.2. For each coalition $S \in \mathbb{I}$, the correspondence F^S, from X to X^S, is continuous with convex and compact values, and $F^S(X) \neq \emptyset$.[9]

When $F^S(x) = X^S$ for every coalition $S \in \mathbb{I}$ and every $x \in X$, we refer to the model as a *standard game* rather than a generalized game.

Remark 2.1. We impose that for each admissible coalition $S \in \mathbb{I}$ the set $F^S(X)$ is non-empty. For coalitions different from the grand coalition or individuals, this assumption is imposed without any loss of generality. Indeed, it is sufficient to replace the set $\mathbb{I}$ by the set $\{S \in \mathbb{I} : \ F^S(X) \neq \emptyset\}$.[10]

We consider the case where agents have (possibly nonordered) preferences displaying externalities (also called interdependent preferences). Formally, each agent i has a preference relation on X which is described by a correspondence P_i from X to X. If $x \in X$ is an

[7]Both notations X^I and X will be used for $\prod_{i \in I} X_i$.

[8]We still denote by π^E the restriction of π^E to any subset X^F where $E \subseteq F \subseteq I$ and for each $i \in I$, the projection $\pi^{\{i\}}$ is denoted by π_i.

[9]The set $F^S(X)$ is the range of the correspondence F^S, defined as $F^S(X) := \bigcup_{x \in X} F^S(x)$.

[10]This is the reason why we do not assume that all coalitions are admissible.

allocation, then $P_i(x)$ represents the set of allocations $y \in X$ that are strictly preferred to x by agent i. We make the standard assumption that preferences are convex.

Assumption 2.3. For each $x \in X$ and each agent i, we have $x \notin \operatorname{co} P_i(x)$.

Our generalized game can be represented by the family

$$\mathcal{G} = \big(L, (X_i, P_i)_{i \in I}, F, (F^S)_{S \in \mathbb{I}}\big).$$

When the feasible sets are degenerate, in the sense that $F^S(x) = X^S$ for all $x \in X$, we obtain the standard definition of a game in normal form. Throughout the paper we will assume that Assumptions 2.1, 2.2 and 2.3 are always satisfied.

An important class of games is constituted by those derived from pure exchange economies, as defined below.

Definition 2.1. A game $\mathcal{G} = \big(L, (X_i, P_i)_{i \in I}, F, (F^S)_{S \in \mathbb{I}}\big)$ is said to be *derived from a standard pure exchange economy (with free-disposal)* if the space L is endowed with a linear order $\geqslant$ such that for each agent i, the strategy set X_i coincides with the cone $L_+ = \{z \in L : z \geqslant 0\}$ and an allocation $(y_i)_{i \in S}$ in L_+^S is feasible for coalition S, i.e., belongs to $F^S(x)$, when

$$\sum_{i \in S} y_i \leqslant \sum_{i \in S} e_i$$

where $e_i \in L_+$ is agent i's initial endowment. In particular, the correspondence F^S takes constant values.

Remark 2.2. If we have a finite number g of commodities, the set L coincides with $\mathbb{R}^g$ and the consumption set of each agent coincides the non-negative cone $\mathbb{R}_+^g$. Under uncertainty with infinitely many states of nature represented by a probability space $(S, \mathcal{S}, \sigma)$, one may choose L to be the space $\mathcal{L}^\infty(S, \mathcal{S}, \sigma)$ of essentially bounded functions and L_+ to be the cone of non-negative functions. We have considered *standard* pure exchange economies for simplicity. The strategy (or consumption) set X_i may be a strict subset of L_+ as it is the case in models with differential information considered in Yannelis (1991a) (see also Podczeck and Yannelis (2008)). For instance, one can restrict agent i's strategies to lie in $\mathcal{L}_+^\infty(S, \mathcal{S}^i, \sigma)$ where $\mathcal{S}^i$ is a sub σ-algebra of $\mathcal{S}$ representing the states agent i can discern ex-post.

In our modeling of preference relations, each agent i ranks allocations in X. Naturally, this modeling encompasses preferences without externalities where agent i only ranks his own strategies in X_i. Indeed, if agent i's preference relation is described by a correspondence $\widehat{P}_i$ defined from X_i to X_i, then we can construct the correspondence P_i as follows:

$$\forall x \in X, \quad P_i(x) = \{(y_k)_{k \in I} \in X \ : \ y_i \in \widehat{P}_i(x_i)\} = \widehat{P}_i(x_i) \times X^{-i}.$$

In that case, we say that agent i's preference relation has *no externalities*.

We also cover the case in which agent i ranks individual strategies while her tastes depend on the current social state. Formally, P_i is induced by a correspondence $\widetilde{P}_i$ from X to X_i via

$$\forall x \in X, \quad P_i(x) = \{(y_k)_{k \in I} \in X : y_i \in \widetilde{P}_i(x)\} = \widetilde{P}_i(x) \times X^{-i}.$$

In this case we say that agent i's preference relation exhibits *weak externalities*. This specification follows the modeling choices in Florenzano (1989) and Lefebvre (2001).

If there exists an ordinal utility function u_i defined from X_i to $[-\infty, \infty)$ such that

$$\forall x = (x_k)_{k \in I} \in X, \quad P_i(x) = \{(y_k)_{k \in I} \in X \ : \ u_i(y_i) > u_i(x_i)\}$$

then agent i's preference relation exhibits no externalities. However, in most game–theoretic models, agent i's preferences depend on the actions of the others through a payoff function $u_i : X \to [-\infty, \infty)$. Let $x = (x_i, x_{-i})$ and define

$$\forall x \in X, \quad P_i(x) := \{y = (y_i, y_{-i}) \in X : u_i(y_i, x_{-i}) > u_i(x_i, x_{-i})\}.$$

Then agent i's preference relation exhibits weak externalities, since only her own component is varied while the others are held fixed. This specification is natural in settings where deviations are unilateral. Because our focus is on cooperative outcomes, it is more appropriate to adopt the coalitional (fully comparative) formulation

$$\forall x \in X, \quad P_i(x) = \{y \in X : u_i(y) > u_i(x)\},$$

which aligns with coalitional deviations and corresponds to the model studied in Scarf (1971).

We consider the following regularity assumption on preferences.

Assumption 2.4. One of the following properties holds.

(i) Each correspondence P_i has an open graph.

(ii) Preferences display weak externalities, i.e., $P_i(x) = \widetilde{P}_i(x) \times X^{-i}$ with $\widetilde{P}_i$ from X to X_i having open lower sections, and each feasibility correspondence F^S is constant.

Condition (i) appears in Scarf (1971), Border (1984), and Kajii (1992). Condition (ii) is included to encompass the existence results of Florenzano (1989) and Lefebvre (2001).

Even with constant feasibility correspondences, assuming that each P_i has open lower sections is not sufficient to deduce that P^S also has open lower sections. To illustrate this, consider the following example.

Example 2.1. Let $I = \{i, j\}$ and $X_i = X_j = \mathbb{R}$. Set

$$F^{\{i\}} = [-1, 1], \qquad F^{\{j\}} = [0, 1], \qquad F^I = [-1, 1] \times [0, 1].$$

All feasibility correspondences are constant, compact, and convex.

Define preferences with open lower sections as follows. For player i, for each $x = (x_i, x_j) \in X$, let

$$P_i(x) := \{(y_i, y_j) \in \mathbb{R}^2 \colon y_j = 0 \text{ or } |x_i| < y_j\}.$$

Fix $y = (a, b) \in X$. If $y \in P_i(x)$, then $b \geqslant 0$. If $b = 0$, then $\{x \colon y \in P_i(x)\} = X$, which is open. If $b > 0$, then $\{x \colon y \in P_i(x)\} = \{(x_i, x_j) \colon |x_i| < b\} = (-b, b) \times \mathbb{R}$, which is open. Hence P_i has open lower sections. For player j, take $P_j(x) := X$, which also has open lower sections.

Consider the singleton coalition $S = \{i\}$. By definition,

$$P^{\{i\}}(x) = \{y_i \in X_i \colon \{y_i\} \times F^{\{j\}} \subseteq P_i(x)\}.$$

Because $F^{\{j\}} = [0, 1]$, the inclusion $\{y_i\} \times [0, 1] \subseteq P_i(x)$ holds if, and only if, $|x_i| < b$ for all

$b \in (0,1]$, i.e., if, and only if, $x_i = 0$. Therefore

$$P^{\{i\}}(x) = \begin{cases} \mathbb{R}, & \text{if } x_i = 0, \\ \emptyset, & \text{if } x_i \neq 0. \end{cases}$$

Fix any $\bar{y}_i \in \mathbb{R}$. Its lower section is

$$\{x \in X \colon \bar{y}_i \in P^{\{i\}}(x)\} = \{(x_i, x_j) \in \mathbb{R}^2 \colon x_i = 0\},$$

which is not open. Hence $P^{\{i\}}$ does not have open lower sections.

3 Core solutions

A core allocation is a feasible social state that is robust to all possible deviations (or blockings) by coalitions. Since the actions of agents outside a blocking coalition influence the welfare of the coalition's members, it is essential to account for how these outsiders may react when defining a core solution. More precisely, when a group of agents forms a coalition to block a given allocation, one must specify the expectations that these agents hold regarding the potential responses of those outside the coalition.

3.1 Weak Blocking Power

Adapting Aumann (1961)'s concept of blocking power to generalized games, Kajii (1992) proposed the following notion of blocking power: a coalition $S \in \mathbb{I}$ is said to α_{K}-block a given social state $x \in F$ if there exists a joint strategy $y = (y_i)_{i \in S}$, feasible for the coalition S, i.e., $y \in F^S(x)$, such that the coalition can ensure a social state strictly preferred by all its members regardless of the strategies chosen by the outsiders. Formally, this means:

$$\big\{(y, w) \ : \ w \in X^{I \setminus S}\big\} \subseteq \bigcap_{i \in S} P_i(x).$$

The $\boldsymbol{\alpha}_{\mathbf{K}}$**-core** is defined as the set of feasible strategies $x \in F$ that cannot be α_{K}-blocked by any coalition.

When forming a blocking coalition S, each agent $i \in S$ adopts a conservative stance, anticipating that outsiders may react in an unrestricted manner–even potentially choosing actions that are detrimental to the coalition.[11] As a result, it is relatively difficult for a coalition to α_{K}-block a feasible strategy, making the α_{K}-core generally large.

Scarf (1971) proved the nonemptiness of the α_{K}-core for standard games in which agents' preferences are represented by continuous and quasi-concave utility functions. This result was later extended by Kajii (1992) to games with possibly nonordered preferences.

3.2 Strong Blocking Power

For generalized games that are not standard,[12] Yannelis (1991b) proposed to increase the blocking power of coalitions by assuming that blocking agents expect the agents outside the coalition to respond with feasible strategies. More precisely, a coalition $S \in \mathbb{I}$ is said to α_{Y}-block a given social state represented by a feasible strategy $x \in F$ if there exists a joint strategy $y \in F^S(x)$, feasible for the coalition S, such that the resulting social state is strictly preferred by all members of the coalition, regardless of any feasible strategies $w \in F^{I\setminus S}(x)$ that the outsiders might adopt, i.e.,

$$\left\{(y, w) \ : \ w \in F^{I\setminus S}(x)\right\} \subseteq \bigcap_{i \in S} P_i(x).$$

The $\boldsymbol{\alpha_{\text{Y}}}$**-core** is then the set of feasible strategies $x \in F$ that cannot be α_{Y}-blocked by any coalition.[13]

If the game is standard (i.e., $F^S = X^S$ for every coalition S), then the α_{Y}-core and the α_{K}-core coincide. However, in generalized games (i.e., $F^S(x) \subsetneq X^S$ for at least one coalition S and one social state $x \in F$), the α_{Y}-core may be strictly smaller than the α_{K}-

[11] In particular, outsiders may retaliate by selecting strategies that are worst from the perspective of agent i's preferences.

[12] In the sense that for at least one coalition S and one social state $x \in X$, the set $F^S(x)$ is different from X^S.

[13] For this definition to be valid, it must be assumed that for each coalition $S \in \mathbb{I}$, the complement $I \setminus S$ is also admissible, i.e.,

$$\forall S \subseteq I, \quad S \in \mathbb{I} \Longrightarrow I \setminus S \in \mathbb{I}.$$

This condition is trivially satisfied when all coalitions are admissible, i.e., $\mathbb{I} = 2^I \setminus \{\emptyset\}$.

core. Indeed, since agents in a blocking coalition assume that outsiders are constrained to feasible responses, it is easier to α_{Y}-block a feasible strategy.

This solution concept is particularly natural in the context of games derived from exchange economies. If a coalition S forms to block a social state x, its members reallocate their initial endowments through strategies (or consumption plans) $(y_i)_{i\in S}$, satisfying

$$\sum_{i\in S} y_i = \sum_{i\in S} e_i.$$

The response $(z_j)_{j\notin S}$ of the counter-coalition must respect the resource constraints of $I \setminus S$, i.e.,

$$\sum_{j\in I\setminus S} z_j = \sum_{j\in I\setminus S} e_j.$$

One may imagine that the agents in $I\setminus S$ form several sub-coalitions and redistribute resources internally. More precisely, the reaction $(z_j)_{j\notin S}$ may be such that

$$I \setminus S = \bigcup_{k\in K} T^k \quad \text{and} \quad \forall k \in K, \quad \sum_{\ell\in T^k} z_\ell = \sum_{\ell\in T^k} e_\ell,$$

where $(T^k)_{k\in K}$ is a finite partition of $I \setminus S$. Nonetheless, the overall response $(z_j)_{j\notin S}$ still belongs to $F^{I\setminus S}$.

Yannelis (1991b) (see also Koutsougeras and Yannelis (1993)) proved that the α_{Y}-core is non-empty for pure exchange economies with at most two agents. For economies with more than two agents, however, the α_{Y}-blocking power may be too strong. Indeed, Holly (1994) provided an example of a pure exchange economy—satisfying standard assumptions—with three agents and an empty α_{Y}-core.

3.3 Intermediate Blocking Power

While proving the nonemptiness of the α_{K}-core, we observed that our arguments actually allow us to establish the nonemptiness of a smaller set. To that end, we introduce a new notion of α-core, which coincides with the α_{Y}-core in the two-agent case, and with the α_{K}-core for standard games. This leads to a general nonemptiness result with the additional and appealing feature of unifying the nonemptiness results for the α_{K}-core and the α_{Y}-core.

Our blocking test is ex ante and evaluates a proposed S-allocation $y^S \in F^S(x)$ against *adverse environments* generated by outsiders' credible individual actions. Fix $x \in F$ and a coalition S. For each outsider $j \notin S$, define

$$F_j(x) := \{\pi_j(z) \,:\, z \in F^E(x),\ E \in \mathbb{I}(j),\ E \neq I\} \quad \text{and} \quad W_j(x) := \operatorname{co}(F_j(x)).$$

Thus $F_j(x)$ collects the actions that j can *credibly secure somewhere* by joining some feasible coalition $E \ni j$ (possibly with $E \cap S \neq \emptyset$ but distinct from the grand coalition). The convexification W_j accounts for randomization or uncertainty over which club forms. Write

$$W^{I\setminus S}(x) := \prod_{j \notin S} W_j(x)$$

for the set of *adverse environments* used to stress-test an S-allocation $y^S \in F^S(x)$.

Definition 3.1. A coalition $S \in \mathbb{I}$ is said to **$\boldsymbol{\alpha}$-block** the feasible joint strategy $x \in F$ if there exists a strategy $y \in F^S(x)$ feasible for the coalition S such that the social state (y, w) is strictly preferred to x by every agent $i \in S$, regardless of the reaction $w = (w_j)_{j \notin S}$ of the counter-coalition, where each w_j belongs to the set $W_j(x)$.[14] The **$\boldsymbol{\alpha}$-core** of the game $\mathcal{G}$ is the set of all feasible strategies $x \in F$ that no coalition α-blocks.

Our notion of α-blocking is consistent with an environment where membership in a deviating coalition is not contractible. After S announces a deviation but before it is implemented, agents may recontract. Outsiders may recruit individual members of S into counter-coalitions that act feasibly. Hence the credible behaviors that outsider j can secure are those she attains by joining some feasible club $E \ni j$, possibly with $E \cap S \neq \emptyset$. Random matching across such clubs and lotteries justify convexification to $W_j := \operatorname{co} F_j$. Allowing $E \cap S \neq \emptyset$ yields a renegotiation-proof intermediate blocking power. It is weaker than granting outsiders arbitrary replies as in the α_{K}-core, and stronger than assuming a single fully coordinated outsider reply as in α_{Y}-core. With two players, α-blocking coincides with α_{Y}-blocking. In standard games without technological or feasibility externalities, $F^E = X^E$ for

[14] If $S = I$, then by convention, (y, w) represents y.

every E, so $F_j = X_j$ and α-blocking reduces to α_{κ}-blocking.

A subtle but important point is that (y^S, w) is *not* required to be jointly feasible. Indeed, if w_j happens to be attainable for j only by joining a counter-coalition E that includes some members of S, then the original y^S would no longer be implementable in that world. This is by design. Our α-blocking is a *robustness* test under partition uncertainty: S asks whether there exists $y^S \in F^S(x)$ such that, *for every* profile $w \in W^{I\setminus S}(x)$ that outsiders might achieve individually somewhere in the economy (even if that requires poaching from S), all members of S strictly prefer y^S to x given the externalities generated by w.

Remark 3.1. A large share of recent work on α-core existence with discontinuous preferences studies environments in which every coalition faces the same feasible action set (equivalently, $F^S(x) = X^S$ for all $S \in \mathbb{I}$), i.e., *standard games* in our terminology. In such settings, the distinctions among blocking powers collapse, and existence arguments do not have to track how the feasibility of deviations changes with the blocking coalition. By contrast, our framework allows coalition-specific feasibility sets $F^S(x) \subsetneq X^S$, which is essential in many economic applications (e.g., technological or resource constraints that depend on the coalition's composition). This heterogeneity directly shapes blocking power and the relative size of the various cores. A natural direction for future work is extending nonemptiness to discontinuous environments *while retaining* $F^S(x) \neq X^S$.

Recall that a feasible strategy x is α-blocked by a coalition $S \in \mathbb{I}$ if there exists a strategy $y \in F^S$ satisfying

$$\big\{(y, w) \ : \ w \in W^{I\setminus S}(x)\big\} \subseteq \bigcap_{i\in S} P_i(x), \tag{3.1}$$

where $W^{I\setminus S}(x) := \prod_{j\notin S} W_j(x)$.

This motivates the following definition. For any coalition $S \in \mathbb{I}$, we define the correspondence P^S from X to X^S as

$$P^S(x) := \left\{ y \in X^S \ : \ \{y\} \times W^{I\setminus S}(x) \subseteq \bigcap_{i\in S} P_i(x) \right\},$$

where $\{y\} \times W^{I\setminus S}(x) := \{(y, w) \ : \ w \in W^{I\setminus S}(x)\}$. In this notation, a feasible joint strategy $x \in F$ belongs to the α-core if, and only if, there does not exist a coalition $S \in \mathbb{I}$ such that

$F^S(x) \cap P^S(x) \neq \emptyset$. Observe that for each coalition $S \in \mathbb{I}$,

$$F^{I\setminus S}(x) \subseteq W^{I\setminus S}(x) \subseteq X^{I\setminus S}.$$

Therefore, the α_{Y}-core is a subset of the α-core, which in turn is a subset of the α_{K}-core. For standard-games—where $F^E(x) = X^E$ for all coalitions $E \in \mathbb{I}$ and all social state $x \in F$—the three core concepts coincide. For games with two agents and convex feasible sets, the α-core coincides with the α_{Y}-core.[15] Indeed, in a two-agent game, the only coalition $E \in \mathbb{I}(i)$ distinct from the grand coalition is the singleton $\{i\}$. Hence, $F_i = F^{\{i\}}$, which implies $W^{I\setminus S} = F^{I\setminus S}$ for any blocking coalition S.

Under the set of assumptions we impose to establish the nonemptiness of the α_{K}-core, we also obtain nonemptiness of the α-core. Therefore, our existence result yields as direct corollaries both the nonemptiness of the α_{K}-core in Kajii (1992) and the nonemptiness of the α_{Y}-core in Yannelis (1991b).

Remark 3.2 (Interpreting W_j in a pure–exchange economy). Assume the commodity space L is a partially ordered real vector space with positive cone L_+, and write $y \geqslant x$ if, and only if, $y - x \in L_+$. Let $(e_i)_{i\in I}$ be the initial endowments (with $e_i \in L_+$), and for each coalition $S \subset I$, set $e(S) := \sum_{i\in S} e_i$. In the pure-exchange case, the S-feasible allocations are the vectors $x^S = (x_i)_{i\in S}$ with $x_i \in L_+$ such that $\sum_{i\in S} x_i \leqslant e(S)$. Hence, if $j \in S$, the j-coordinate projection is the order-interval $[0, e(S)]$. We deduce that

$$F_j = \bigcup_{\substack{S\subset I,\ S\neq I \\ j\in S}} [0, e(S)].$$

Define the (finite) convex hull of the relevant endpoints

$$A_j = \mathrm{co}\{e(S) \colon S \subset I, S \neq I, \text{ and } S \ni j\}.$$

If, in addition, L has the Riesz decomposition property, then the set W_j of possible actions

[15] In a two-agent game $\{i, j\}$, we say the feasible sets are convex if both $F^{\{i\}}(x)$ and $F^{\{j\}}(x)$ are convex for all social states $x \in F$. Note that $F^I = F$ is already assumed to be convex. In games derived from standard pure exchange economies, feasible sets are always convex.

of outsider j is

$$W_j = L_+ \bigcap [A_j - L_+].$$

In words, W_j is exactly the set of nonnegative bundles that are coordinatewise no larger than some convex combination of coalition endowments $e(S)$ over coalitions $S \ni j$, $S \neq I$. Finally, note that $A_j \subseteq [0, e(I)]$, hence $W_j \subseteq [0, e(I)]$.

4 Non-emptiness of the α-core

The main contribution of this paper is to show that, in addition to Assumptions 2.1–2.4, a balancedness condition on the feasibility correspondences suffices to ensure the nonemptiness of the α-core. The notion of a *balanced* n-person game was introduced by Bondareva (1962) and Shapley (1965) in the setting of transferable utility. In the nontransferable utility setting, balancedness was employed by Scarf (1967) for games in characteristic form and by Scarf (1971) for games in normal form. We now give the formal definition.

4.1 Balancedness

Let Δ denote the set of weight vectors $\lambda = (\lambda_S)_{S \in \mathbb{I}} \in \mathbb{R}^{\mathbb{I}}_+$ that satisfy the condition

$$\forall i \in I, \quad \sum_{S \in \mathbb{I}(i)} \lambda_S = 1.$$

An element $\lambda \in \Delta$ is called a *balancing weight*, and the associated family of coalitions $\{S \in \mathbb{I} : \lambda_S > 0\}$ is said to be *balanced*.

Remark 4.1. Observe that the set Δ is always non-empty. For example, consider the vector $\kappa = (\kappa_S)_{S \in \mathbb{I}}$ defined by $\kappa_S = 1$ if S is a singleton and $\kappa_S = 0$ otherwise. Since all singletons $\{i\}$ belong to $\mathbb{I}$, we have $\kappa \in \Delta$.

We now recall the definition of a balanced game.[16]

[16] Several generalizations of this concept appear in the literature, including π-balancedness Billera (1970), Π-balancedness Predtetchinski and Herings (2004), and payoff-dependent balancedness Bonnisseau and Iehlé (2007).

Definition 4.1. The game $\mathcal{G} = \big(L, (X_i, P_i)_{i\in I}, F, (F^S)_{S\in\mathbb{I}}\big)$ is said to be **balanced** if, for every balancing weight $\lambda \in \Delta$ and every social state $x \in F$, the following holds: If $y^S \in F^S(x)$ is a feasible strategy for each coalition $S \in \mathbb{I}$ with $\lambda_S > 0$, then the strategy $z = (z_i)_{i\in I}$ defined by

$$\forall i \in I, \quad z_i := \sum_{S\in\mathbb{I}(i)} \lambda_S y_i^S \tag{4.1}$$

is a feasible social state, i.e., $z \in F$.

It is straightforward to verify that games derived from pure exchange economies always satisfy this balancedness condition.

Remark 4.2. Let $(x_i)_{i\in I} \in X$ be a profile of individually feasible strategies, i.e., $x_i \in F^{\{i\}}(x)$ for each agent $i \in I$. If the game $\mathcal{G}$ is balanced, then the resulting social state $x = (x_i)_{i\in I}$ is feasible for the grand coalition, i.e., $x \in F$. To see this, consider the balancing weight κ introduced in Remark 4.1 and apply Equation (4.1).

Before presenting our main contribution, we briefly review the existing core existence results in the literature.

4.2 The Literature

Agent i's preference relation is said to be *ordered* if there exists a utility function $u_i : X \to [-\infty, \infty)$ such that a joint strategy y is strictly preferred to another strategy x if, and only if, $u_i(y) > u_i(x)$. Scarf (1971) proved the following existence result:

Theorem 4.1 (Scarf (1971))**.** *Consider a balanced generalized game whose strategy sets and feasible correspondences satisfy Assumptions 2.1 and 2.2. If agents' preference relations are represented by continuous and quasi-concave utility functions, then the α_{K}-core of a balanced game is non-empty.*

Strictly speaking, Scarf (1971) proved this result for strategy sets that are subsets of finite-dimensional Euclidean spaces and for standard (not generalized) games. However, his arguments can be straightforwardly extended to more general Hausdorff topological vector space.

Kajii (1992) explored whether the assumption of ordered preferences could be relaxed. This question had previously been addressed by Border (1984) for games without externalities, and by Florenzano (1989) in settings with weak externalities.[17] Both results pertain to the standard core rather than the α-core, as they do not account for externalities.

To our knowledge, Kajii (1992) provides the only existence result for the α-core in nontransferable utility games with nonordered preferences and externalities. Kajii generalized Border's approach by introducing a pseudo-utility function $u_i : X \times X \to \mathbb{R}_+$, where $u_i(x, y)$ is defined as the distance from the pair (x, y) to the complement of the graph of the correspondence P_i. If P_i has an open graph, then $u_i(x, y) > 0$ if, and only if, y is strictly preferred to x. To apply Scarf's theorem, one needs $y \mapsto u_i(x, y)$ to be quasi-concave, which holds if the distance is induced by a norm. This leads to the following result:

Theorem 4.2 (Kajii (1992))**.** *Consider a balanced generalized game whose strategy sets and feasible correspondences satisfy Assumptions 2.1 and 2.2. Assume, moreover, that the topology of the strategy vector space L is norm-induced. If each preference correspondence P_i satisfies Assumption 2.3 and condition (i) of 2.4 (i.e., P_i has an open graph), then the α_{K}-core is non-empty.*

This is an important contribution, but one may find the restriction to normed strategy spaces unsatisfactory. As Kajii (1992) notes, the main limitation lies in the norm-compactness assumption. From a theoretical standpoint, his result does not generalize those of Scarf (1971) or Florenzano (1989), both of which apply to general Hausdorff topological vector spaces. From a practical perspective, norm-compactness imposes a severe restriction in infinite-dimensional strategy spaces. This issue is well documented in Mas-Colell and Zame (1991), which shows that in exchange economies, the set of feasible trades is typically weakly compact but not norm-compact.[18]

Although Kajii (1992) provides no counterexamples, one might believe that the price for incorporating nonordered preferences and externalities is to restrict analysis to normed topologies. This paper demonstrates that such a trade-off is not necessary. We show that nonemptiness of the α-core can be achieved for general preference relations—nonordered

[17] Florenzano (1989) actually proves the nonemptiness of the core in a production economy. However, her arguments can be adapted to games.

[18] We also provide illustrative examples in Section 6.

and exhibiting externalities—under very general topological assumptions. In particular, our results extend and unify those of Scarf (1971), Florenzano (1989), and Kajii (1992).

Relation to recent work. Several recent papers establish nonemptiness of the α–core under discontinuous and/or nonordered preferences in strategic (normal–form) games with a common feasible set; see Uyanik (2015), Basile and Scalzo (2020) and Scalzo (2022) for models with discontinuous preferences, and Yang and Yuan (2019), Yang (2017, 2018, 2020), Yang and Zhang (2021), Yang and Song (2022), and Song and Guo (2022) for games with nonordered preferences and/or infinitely many players. These contributions, however, do not allow for coalition–specific feasibility sets F^S in finite–player nontransferable utility games, and thus do not address the differences in blocking power that arise when feasibility depends on the deviating coalition. In contrast, Kajii (1992) provides an existence result for the α–core in generalized games with nonordered preferences *and* externalities *and* coalition–dependent feasibility.

4.3 The Main Result

We are now ready to state the main result of the paper, whose proof is given in Section 7.

Main Theorem. *Consider a balanced generalized game satisfying Assumptions 2.1–2.4, i.e.:*

(1) For each agent i, the strategy set X_i is a non-empty and convex subset of a Hausdorff topological vector space L.

(2) For each coalition $S \in \mathbb{I}$, the correspondence F^S, from X to X^S, is continuous with convex and compact values, and $F^S(X) \neq \emptyset$.

(3) For each $x \in X$ and each agent i, we have $x \notin \operatorname{co} P_i(x)$.

(4) One of the following properties holds.

(i) For each agent i, the correspondence P_i has an open graph.

(ii) For each agent i, preferences display weak externalities, i.e., $P_i(x) = \widetilde{P}_i(x) \times X^{-i}$ with $\widetilde{P}_i$ from X to X_i having open lower sections, and each feasibility correspondence F^S is constant.

Then its α-core is non-empty.

This nonemptiness result generalizes and unifies the findings of Scarf (1971), Florenzano (1989) and Lefebvre (2001), and Kajii (1992). More importantly, it answers the open question raised in (Kajii 1992, Section 4) by allowing arbitrary linear topologies on the strategy space. We provide two examples in Section 6 where our result applies, but existing results do not.

Our nonemptiness result contributes to the literature in another important way: it establishes the existence of a smaller core. Our α-core coincides with the α_{Υ}-core when there are at most two agents and the feasible sets are convex. As a result, the Main Theorem yields, as a special case, the nonemptiness result of Yannelis (1991b) (see also Koutsougeras and Yannelis (1993)).

Remark 4.3. Recently, Bonnisseau and Iehlé (2007) proved the nonemptiness of the core for NTU games under a condition called payoff-dependent balancedness, involving transfer rate mappings (see also Predtetchinski and Herings (2004) for related results). Their framework allows them to recover Kajii's nonemptiness theorem. However, their approach also relies on constructing pseudo-utility functions (as in Border (1984) and Kajii (1992)), and crucially, they require the strategy space to be a normed vector space in order to obtain quasi-concavity.

Remark 4.4. Our existence proof assumes compactness and convexity of feasibility sets, balancedness, and open graphs of the preference correspondences. These hypotheses are standard and directly verifiable in applications with continuous preferences and convex technologies. They are not intrinsic to the economics of our intermediate α–blocking power. In particular, continuity can be replaced by security/approximation conditions familiar from the literature on discontinuous games (e.g., better–reply security or small payoff perturbations), without altering the underlying economic content. Recently, several advances were made on establishing infinite dimensional equilibrium existence theorems with discontinuous preferences (see, for example, Podczeck and Yannelis (2024) and Khan et al. (2025)). A natural open question is whether similar results can be established for the existence of the α-core with an infinite-dimensional strategy space and discontinuous preferences. We leave a full treatment of these extensions to future research.

5 An Example Differentiating Core Concepts

Consider a pure exchange economy with three agents $I = \{1, 2, 3\}$ and a single good. Each agent is endowed with one unit of the good, so the aggregate endowment is 3 units. Agents choose their consumption levels in the set $X_i = [0, 3]$. The feasible consumption sets are $F^{\{i\}} = [0, 1]$ for singleton coalitions and $F^{\{i,j\}} = \{(x_i, x_j) \in \mathbb{R}^2_+ : x_i + x_j \leqslant 2\}$ for any pair of distinct agents $i \neq j$. Consequently, the set of feasible outsider responses is $W_i = [0, 2]$ for each agent i.

The utility functions are defined by

$$u_i(x) = x_i - \max\{x_j, x_k\}, \quad \text{where } \{i, j, k\} = I,$$

capturing negative externalities from the consumption of others.

Proposition 5.1. In this pure exchange economy, the three core concepts are pairwise distinct. Specifically,

$$\alpha_{\mathrm{Y}}\text{-core} \subsetneq \alpha\text{-core} \subsetneq \alpha_{\mathrm{K}}\text{-core}.$$

The proposition follows from the claims established below, where we exhibit allocations that belong to one core but not another, thereby demonstrating the strict inclusions.

Claim 5.1. The allocation $x = (0, 3/2, 3/2)$ belongs to the α_{K}-core but not the α-core.

Proof. The allocation $x = (0, 3/2, 3/2)$, which yields the utility profile

$$u_1(x) = -3/2, \quad u_2(x) = 0, \quad \text{and} \quad u_3(x) = 0.$$

We begin by showing that agent 1 can α-block the allocation x by consuming his endowment, i.e., choosing $y_1 = 1$. For any response $(w_2, w_3) \in W_2 \times W_3 = [0, 2]^2$, we have

$$u_1(y_1, w_2, w_3) = 1 - \max\{w_2, w_3\} \geqslant 1 - 2 = -1 > -3/2 = u_1(x),$$

so agent 1 strictly prefers the deviation under all feasible reactions by the outsiders. Hence, $x \notin \alpha$-core.

We now turn to proving that $x \in \alpha_{\text{K}}$-core. We begin by showing that no singleton coalition can α_{K}-block x.

First, consider agent 1. For any feasible choice $y_1 \in F^{\{1\}} = [0,1]$, a possible response from the counter-coalition is $(z_2, z_3) = (3,3)$, which is allowed under α_{K}-blocking. Then,

$$u_1(y_1, z_2, z_3) = y_1 - \max\{z_2, z_3\} = y_1 - 3 \leqslant -2 < -3/2 = u_1(x).$$

Next, consider agent 2. For any feasible choice $y_2 \in F^{\{2\}} = [0,1]$, a valid response from the outsiders is $(z_1, z_3) = (3,3)$, yielding

$$u_2(z_1, y_2, z_3) = y_2 - \max\{z_1, z_3\} = y_2 - 3 \leqslant -2 < 0 = u_2(x).$$

Hence, agent 2 cannot block x either. By symmetry, the same argument applies to agent 3. Therefore, no singleton coalition can α_{K}-block x.

Let us now analyze potential deviations by coalitions consisting of two agents. First, consider coalition $\{1,2\}$. For any feasible consumption plan $(y_1, y_2) \in \mathbb{R}^2_+$ satisfying $y_1 + y_2 \leqslant 2$, a valid response from the outsider (agent 3) is $z_3 = 3$. The resulting utilities are:

$$u_1(y_1, y_2, z_3) = y_1 - \max\{y_2, 3\} = y_1 - 3 \quad \text{and} \quad u_2(y_1, y_2, z_3) = y_2 - \max\{y_1, 3\} = y_2 - 3.$$

Suppose, for contradiction, that both agents strictly prefer the deviation: $u_1(y_1, y_2, z_3) > u_1(x)$ and $u_2(y_1, y_2, z_3) > u_2(x)$. Adding these inequalities yields

$$y_1 + y_2 - 6 > -3/2,$$

contradicting the feasibility condition $y_1 + y_2 \leqslant 2$. Hence, coalition $\{1,2\}$ cannot α_{K}-block x. By symmetry, coalition $\{1,3\}$ cannot block x either.

Next, consider coalition $\{2,3\}$. For any feasible pair $(y_2, y_3) \in \mathbb{R}^2_+$ with $y_2 + y_3 \leqslant 2$, a valid response from agent 1 is $z_1 = 3$. Then:

$$u_2(z_1, y_2, y_3) = y_2 - \max\{y_3, 3\} = y_2 - 3 \quad \text{and} \quad u_3(z_1, y_2, y_3) = y_3 - \max\{y_2, 3\} = y_3 - 3.$$

Suppose both agents strictly improve: $u_2(z_1, y_2, y_3) > u_2(x)$ and $u_3(z_1, y_2, y_3) > u_3(x)$.

Adding these inequalities yields

$$y_2 + y_3 - 6 > 0$$

which again contradicts the feasibility condition $y_2 + y_3 \leqslant 2$. Therefore, coalition $\{2, 3\}$ also cannot α_{K}-block x.

It remains to verify that the grand coalition $I = \{1, 2, 3\}$ cannot α_{K}-block x. Suppose, for contradiction, that there exists a feasible allocation $y = (y_1, y_2, y_3) \in \mathbb{R}^3_+$ with $y_1 + y_2 + y_3 = 3$ such that each agent strictly prefers y to x; that is, $u_i(y) > u_i(x)$ for every $i \in I$. Since $u_2(x) = u_3(x) = 0$, we must have

$$y_2 > \max\{y_1, y_3\} \quad \text{and} \quad y_3 > \max\{y_1, y_2\}.$$

But these two inequalities cannot hold simultaneously. The first implies $y_2 > y_3$, and the second implies $y_3 > y_2$, which is a contradiction. Therefore, no such allocation y exists, and the grand coalition cannot α_{K}-block x. We conclude that $x \in \alpha_{\mathrm{K}}$-core. □

The game induced by this exchange economy satisfies all the assumptions of the Main Theorem. Consequently, the α-core is guaranteed to be nonempty as illustrated by the following claim.

Claim 5.2. The symmetric allocation $x = (1, 1, 1)$ lies within the α-core of the economy.

Proof. This allocation is feasible, as each agent consumes one unit and the total endowment is 3 units. The utility levels are

$$u_i(x) = 1 - \max\{1, 1\} = 0 \quad \text{for all } i \in I.$$

We now verify that no coalition can α-block this allocation.

Let $S = \{i\}$ for some agent i. Any feasible deviation satisfies $y_i \in [0, 1]$, and the counter-coalition may respond with $(w_j, w_k) \in [0, 2] \times [0, 2]$, yielding

$$u_i(y_i, w_j, w_k) = y_i - \max\{w_j, w_k\} \leqslant 1 - 2 = -1 < u_i(x) = 0.$$

Hence, no singleton coalition can α-block x.

Consider $S = \{i, j\}$ and let k be the remaining agent. The coalition is restricted to $(y_i, y_j) \geqslant 0$ with $y_i + y_j \leqslant 2$, and the outsider may respond with $w_k \in W_k = [0, 2]$. Then, choosing $w_k = 2$,

$$u_i(y_i, y_j, w_k) = y_i - \max\{y_j, w_k\} \leqslant y_i - 2 \leqslant -1 < u_i(x),$$

and by symmetry $u_j(y_i, y_j, w_k) < u_j(x)$. So neither agent strictly prefers the deviation, and no two-agent coalition can α-block x.

Let $S = I$. Any feasible allocation $y = (y_1, y_2, y_3) \geqslant 0$ satisfies $y_1 + y_2 + y_3 \leqslant 3$. Suppose, for contradiction, that $u_i(y) > u_i(x)$ for all i. Since $u_i(x) = 0$, we get

$$y_i > \max\{y_j, y_k\} \quad \text{for all } i \in I.$$

This system is inconsistent: for example, $y_1 > \max\{y_2, y_3\}$ implies $y_1 > y_2$ and $y_1 > y_3$, but similarly $y_2 > y_1$ and $y_3 > y_1$ would be required, leading to a contradiction. Hence, the grand coalition cannot α-block x.

We conclude that $x = (1, 1, 1)$ belongs to the α-core. □

Finally, we show that the α_{Y}-core is a strict subset of the α-core.

Claim 5.3. The allocation $x = (1/2, 5/4, 5/4)$ belongs to the α-core, but not the α_{Y}-core.

Proof. We compute

$$u_1(x) = -3/4, \quad u_2(x) = 0, \quad \text{and} \quad u_3(x) = 0.$$

We begin by showing that coalition $\{1, 2\}$ can α_{Y}-block the allocation x by choosing the allocation $(y_1, y_2) = (3/4, 5/4)$. As agent 3 can respond by choosing $w_3 \in [0, 1]$, we have

$$u_1(y_1, y_2, w_3) = 3/4 - \max\{5/4, w_3\} = -1/2 > -3/4 = u_1(x),$$

and

$$u_2(y_1, y_2, w_3) = 5/4 - \max\{3/4, w_3\} \geqslant 5/4 - \max\{3/4, 1\} = 1/4 > 0 = u_2(x),$$

so coalition $\{1, 2\}$ successfully α_{Y}-blocks the allocation x.

We now verify that no coalition can α-block this allocation.

Let $S = \{i\}$ for some agent i. Any feasible deviation satisfies $y_i \in [0, 1]$, and the counter-coalition may respond with $(w_j, w_k) \in [0, 2] \times [0, 2]$, yielding

$$u_i(y_i, w_j, w_k) = y_i - \max\{w_j, w_k\} \leqslant 1 - 2 = -1 < u_i(x),$$

as $u_i(x) \in \{-3/4, 0\}$. Hence, no singleton coalition can α-block x.

Consider $S = \{i, j\}$ and let k be the remainder agent. The coalition is restricted to $(y_i, y_j) \geqslant 0$ with $y_i + y_j \leqslant 2$, and the outsider can respond with $w_k = 2$. Then,

$$u_i(y_i, w_j, w_k) = y_i - \max\{w_j, w_k\} \leqslant y_i - 2 \leqslant 0.$$

As $0 \in \{u_i(x), u_j(x)\}$, at least one of the agents does not strictly improve, and no two-agent coalition can α-block x.

Let $S = I$. Fix a feasible allocation $y = (y_1, y_2, y_3)$ and suppose, for contradiction, that $u_i(y) > u_i(x)$ for all i. Since $u_2(x) = u_3(x) = 0$, we get

$$y_2 > \max\{y_1, y_3\} \quad \text{and} \quad y_3 > \max\{y_1, y_2\}.$$

This system is inconsistent as it implies $y_2 > y_3$ and $y_3 > y_2$. Hence, the grand coalition cannot α-block x.

We conclude that $x = (1/2, 5/4, 5/4)$ belongs to the α-core. □

6 Applications

We show that the hypotheses of the Main Theorem are easily met in two illustrative economic environments. These examples also demonstrate that the compactness assumptions naturally call for non-normable topologies, thereby clarifying the limits of Kajii (1992)'s result.

6.1 Mixed Strategies Over Infinitely Many Pure Actions

Let $(\mathbb{A}, \tau)$ be a compact topological space. Consider a (non-generalized) game in which each agent i selects a mixed strategy σ_i from the set X_i of Borel probability measures over a closed subset $A_i \subseteq \mathbb{A}$ of pure actions. The strategy space is $\mathbb{M}$, the vector space of Borel signed measures on $\mathbb{A}$. Equipped with the weak topology $\sigma(\mathbb{M}, \mathbb{C})$—where $\mathbb{C}$ denotes the space of τ-continuous real-valued functions on $\mathbb{A}$—each X_i is compact.[19]

In many models, agent i's utility from a joint strategy $\sigma = (\sigma_i)_{i \in I}$ is given by the expected value

$$\mathbb{E}^{\sigma}[v_i] := \int_A v_i(a)\sigma(da),$$

for some continuous felicity function $v_i : A \to \mathbb{R}$ defined on $A = \prod_{i \in I} A_i$. Here, we propose an alternative modeling of preferences: agent i represents a group K_i of individuals, each indexed by $k \in K_i$ and endowed with a continuous felicity function $v_i(k, \cdot) : A \to \mathbb{R}$. Group preferences are aggregated via a family $\mathcal{K}_i$ of non-empty subsets of K_i.

Definition 6.1. Agent i is said to have **voting preferences** represented by $\mathcal{K}_i$ if a mixed strategy profile η is strictly preferred to another profile σ whenever there exists $\kappa_i \in \mathcal{K}_i$ such that

$$\forall k \in \kappa_i, \quad \mathbb{E}^{\eta}[v_i(k)] > \mathbb{E}^{\sigma}[v_i(k)].$$

That is, the set of strictly preferred profiles is

$$P_i(\sigma) := \bigcup_{\kappa_i \in \mathcal{K}_i} \bigcap_{k \in \kappa_i} P_{i,k}(\sigma),$$

where $P_{i,k}(\sigma)$ contains all profiles η for which individual k strictly prefers η to σ. If $\mathcal{K}_i = \{K_i\}$, this corresponds to a unanimity rule.

Theorem 6.1. *Assume that each K_i is finite and the voting rule $\mathcal{K}_i$ satisfies*

$$\bigcap_{\kappa_i \in \mathcal{K}_i} \kappa_i \neq \emptyset. \tag{6.1}$$

Then the α-core of the game with voting preferences $(\mathcal{K}_i)_{i \in I}$ is non-empty.

[19] This topology is also known as the weak-star topology or the topology of convergence in distribution.

Let $\pi_i^\star$ denote the set of individuals who belong to all $\kappa_i \in \mathcal{K}_i$. Condition (6.1) ensures that $\pi_i^\star \neq \emptyset$—that is, there is at least one individual whose approval is required for any group κ_i.

Example 6.1. A typical example is $\mathcal{K}_i(k_i^\star, \alpha)$, the family of subsets $\kappa_i \subseteq K_i$ that include a fixed individual $k_i^\star$ and contain at least a proportion α of the total, i.e., $|\kappa_i| \geqslant \alpha |K_i|$. Here, the decision maker prefers η to σ only if it is favored by a sufficiently large subset that includes $k_i^\star$.[20]

If η is strictly preferred to σ by agent i, this means that there exists a set $\kappa_i \in \mathcal{K}_i$ such that every $k \in \kappa_i$ strictly prefers η to σ. Importantly, the set κ_i may vary with the pair (η, σ). Therefore, the preference relation induced by voting is generally neither transitive nor complete.[21] Hence, we cannot apply Scarf (1971). Moreover, since the weak topology $\sigma(\mathbb{M}, \mathbb{C})$ is not normable when A_i is infinite, we also cannot invoke Kajii (1992).

Proof of Theorem 6.1. We verify that all the assumptions of the Main Theorem are satisfied. Assumptions 2.1 and 2.2 are trivially satisfied. Since the game is standard (i.e., $F^S = X^S$ for all S), it is automatically balanced.

To prove that Assumption 2.4 is satisfied, we show that each P_i has an open graph in the weak product topology $\sigma(\mathbb{M}, \mathbb{C})^I$. We have

$$\operatorname{gph} P_i = \bigcup_{\kappa_i \in \mathcal{K}_i} \bigcap_{k \in \kappa_i} \operatorname{gph} P_{i,k},$$

where $\operatorname{gph} P_{i,k} := \{(\eta, \sigma) \,:\, \mathbb{E}^\eta[v_i(k)] > \mathbb{E}^\sigma[v_i(k)]\}$. Since $v_i(k)$ is continuous, each $\operatorname{gph} P_{i,k}$ is open in the product weak topology. Because K_i is finite, so are the intersections, and hence $\operatorname{gph} P_i$ is open.

Finally, we verify Assumption 2.3. Suppose by contradiction that there exists a finite set $(\eta^\ell)_{\ell \in L}$ and weights $(x^\ell)_{\ell \in L}$ such that

$$\forall \ell \in L, \quad \eta^\ell \in P_i(\sigma), \quad \text{and} \quad \sigma = \sum_{\ell \in L} x^\ell \eta^\ell.$$

[20] The decision maker could also be modeled as one of the individuals.

[21] This holds, for instance, in Example 6.1 with $|K_i| = 3$ and $\alpha = 2/3$.

Let $k_i^\star$ be an individual in the intersection (6.1). Then,

$$\mathbb{E}^\sigma[v_i(k_i^\star)] = \sum_{\ell \in L} x^\ell \mathbb{E}^{\eta^\ell}[v_i(k_i^\star)] > \mathbb{E}^\sigma[v_i(k_i^\star)],$$

a contradiction. Therefore, the game satisfies all conditions of the Main Theorem. □

6.2 Uncertainty With Infinitely Many States of Nature

Let S be a countable set of states of nature. Denote by $B(n)$ the space of bounded functions from S to $\mathbb{R}^n$. For each $s \in S$, let

$$\mathcal{G}(s) = \big(L, (X_i, u_i)_{i\in I}, F(s), (F^E(s))_{E\in\mathbb{I}}\big)$$

be a generalized balanced game satisfying: (i) $L = \mathbb{R}^n$, (ii) X_i is a non-empty convex subset of L, (iii) $u_i : X \to \mathbb{R}$ is continuous, concave, and bounded, (iv) $F^E(s)$ is non-empty and compact for each E, and (v) $F(s)$ is convex. An example is a game from a pure exchange economy $\mathcal{E}(s)$ with initial endowments $(e_i(s))_{i\in I}$.

In the associated **ex-ante game**, each agent i chooses a bounded, state-contingent strategy $x_i \in B(n)$ such that $x_i(s) \in X_i$ for every s. Let $\mathcal{X}_i$ be the corresponding strategy set and $\mathcal{L} = B(n)$ the underlying space. The feasible set $\mathcal{F}^E$ consists of profiles $x^E = (x_i)_{i\in E} \in \mathcal{L}^E$ such that $x^E(s) \in F^E(s)$ for all s.

As in Bewley (2002) and Rigotti and Shannon (2005), agent i has a **Knightian preference** defined by a set $Q_i \subseteq \text{Prob}(S)$. Given two profiles x and y, $y \in P_i(x)$ if

$$\forall q \in Q_i, \quad \int_S u_i(y(s))q(ds) > \int_S u_i(x(s))q(ds).$$

Then the ex-ante game is given by

$$\mathcal{G} = \big(\mathcal{L}, (\mathcal{X}_i, P_i)_{i\in I}, \mathcal{F}, (\mathcal{F}^E)_{E\in\mathbb{I}}\big).$$

Theorem 6.2. *Suppose each $Q_i \subseteq \text{Prob}(S)$ is non-empty, convex, and compact in the weak-*

star topology.[22] *Then the α-core of the ex-ante game $\mathcal{G}$ is non-empty.*

Proof of Theorem 6.2. We verify the conditions of the Main Theorem. Each state game $\mathcal{G}(s)$ is balanced, so the ex-ante game is also balanced. Assumptions 2.1 and 2.3 hold. With $B(n)$ endowed with the product topology, Assumption 2.2 follows from Tychonoff's theorem.

Fix $i \in I$. To verify Assumption 2.4, we prove P_i has an open graph. Consider $x, y \in \mathcal{X}$ with $y \in P_i(x)$. Define $\varphi(q) := \int_S [u_i(y(s)) - u_i(x(s))] q(ds)$. Then φ is continuous in the weak-star topology. Since Q_i is compact, there exists $\varepsilon > 0$ such that

$$\inf_{q \in Q_i} \varphi(q) \geqslant 2\varepsilon.$$

By tightness, there exists a finite $S(\varepsilon) \subseteq S$ such that for all $q \in Q_i$,

$$q(S \setminus S(\varepsilon)) \leqslant \frac{\varepsilon}{5M},$$

where M bounds $|u_i|$. By continuity, we can find neighborhoods V_x, V_y around x, y on $S(\varepsilon)$ such that for all $(\widetilde{x}, \widetilde{y})$ in $W_x \times W_y$,

$$\int_S u_i(\widetilde{y}(s)) q(ds) > \int_S u_i(\widetilde{x}(s)) q(ds) \quad \text{for all } q \in Q_i.$$

Hence, P_i has an open graph. ∎

7 Proof of the Main Theorem

The proof of the Main Theorem draws inspiration from Proposition 1 and Proposition 2 in Florenzano (1989) and Theorem 2.1 in Lefebvre (2001). Our framework is more general than those considered in Florenzano (1989) and Lefebvre (2001), as we allow for externalities in preferences. A crucial difference from Florenzano (1989) lies in our use of a result on the representation of balanced collections established in (Scarf 1971, pp. 178–179). Our main technical contribution lies in combining the techniques of Scarf (1971) and Florenzano (1989). We divide the proof into two steps: First, we handle the finite-dimensional case; the general

[22] This is $\sigma(\text{Prob}(S), B)$, where $B = B(1)$ is the space of bounded real-valued functions on S.

result is then derived using a Bewley-type limit argument.

7.1 The Finite-Dimensional Case

The fixed-point theorem we apply (see Lemma A.1 in Appendix A.1) is valid in finite-dimensional spaces, motivating our initial focus on that setting.

Proposition 7.1. Let $\mathcal{G} = \big(L, (X_i, P_i)_{i\in I}, F, (F^S)_{S\in\mathbb{I}}\big)$ be a balanced game satisfying Assumptions 2.1, 2.2, 2.3, and 2.4, If L is finite-dimensional, then $\mathcal{G}$ has a non-empty α-core.

Proof of Proposition 7.1. Let

$$Z := \prod_{S\in\mathbb{I}} F^S(X)$$

which is non-empty, compact, and convex.[23] Recall from Section 4 that Δ denotes the set of balancing weights. For $(z,\lambda) \in Z \times \Delta$, where $z = (z^S)_{S\in\mathbb{I}}$ and $z^S = (z_i^S)_{i\in S}$, define

$$\theta(z,\lambda) := (y_i)_{i\in I}, \quad \text{where} \quad y_i := \sum_{S\in\mathbb{I}(i)} \lambda_S z_i^S.$$

Because the game is balanced, $\theta(Z\times\Delta) \subseteq \operatorname{co} F = F$.

For each $x \in F$, define

$$\varphi(x) := \prod_{S\in\mathbb{I}} \varphi^S(x), \quad \text{where } \varphi^S(x) := \operatorname{co}\big[F^S(x) \cap P^S(x)\big].$$

Set

$$\mathbb{I}(x) := \big\{S \in \mathbb{I} \;:\; \varphi^S(x) \neq \emptyset\big\},$$

and let

$$\Sigma := \left\{\mu = (\mu_S)_{S\in\mathbb{I}} \in \mathbb{R}_+^{\mathbb{I}} \;:\; \sum_{S\in\mathbb{I}} \mu_S = 1\right\}.$$

For $(x,\mu) \in F \times \Sigma$, define

$$\psi(\mu) := \arg\max\{\mu\cdot\lambda \;:\; \lambda \in \Delta\} \quad \text{and} \quad \xi(x) := \{\nu \in \Sigma \;:\; \nu_S = 0 \text{ for all } S \notin \mathbb{I}(x)\}.$$

[23] Recall that the set $F^S(X)$ is the range of the correspondence F^S, defined as $F^S(X) := \bigcup_{x\in X} F^S(x)$. As each correspondence $F^S : X \to X^S$ is continuous and X is compact, we get that $F^S(X)$ is compact.

Let

$$K := F \times Z \times \Delta \times \Sigma,$$

which is a non-empty, convex, compact subset of a finite-dimensional space. Define the correspondence $\chi : K \to K$ as follows:

$$\chi(x, z, \lambda, \mu) := \{\theta(z, \lambda)\} \times \varphi(x) \times \psi(\mu) \times \xi(x).$$

We verify that χ satisfies the hypotheses of Lemma A.1. Since θ is continuous, the correspondence $(z, \lambda) \mapsto \{\theta(z, \lambda)\}$ is continuous with compact, convex and nonempty values. By Proposition A.2 in appendix, each correspondence $x \mapsto F^S(x) \cap P^S(x)$ is lower semicontinuous. Convexification preserves lower semicontinuity, hence each φ^S is lower semicontinuous. In particular, for every $x \in F$ there exists an open neighborhood W of x such that $\mathbb{I}(x) \subseteq \mathbb{I}(x')$ for all $x' \in W$.[24] Consequently, ξ has open lower sections and is therefore lower semicontinuous. By Berge's Maximum Theorem, ψ is nonempty, compact–valued, convex–valued, and upper semicontinuous.

By Lemma A.1, there exists $(\overline{x}, \overline{z}, \overline{\lambda}, \overline{\mu}) \in K$ such that:

$$\overline{x} = \theta(\overline{z}, \overline{\lambda}), \tag{7.1}$$

$$\forall S \in \mathbb{I}, \quad \overline{z}^S \in \varphi^S(\overline{x}) \quad \text{or} \quad \varphi^S(\overline{x}) = \emptyset, \tag{7.2}$$

$$\overline{\lambda} \in \psi(\overline{\mu}), \tag{7.3}$$

$$\overline{\mu} \in \xi(\overline{x}) \quad \text{or} \quad \mathbb{I}(\overline{x}) = \emptyset. \tag{7.4}$$

We now prove that $\overline{x}$ belongs to the α-core. If $\mathbb{I}(\overline{x}) = \emptyset$, then $\varphi^S(\overline{x}) = \emptyset$ for all S, so $\overline{x}$ is in the α-core.

Assume for contradiction that $\mathbb{I}(\overline{x}) \neq \emptyset$. By Claim 7.1 (proven in Appendix A.3, see also Lefebvre (2001)):

Claim 7.1. There exists $i_0 \in I$ such that for all $S \notin \mathbb{I}(\overline{x})$, if $i_0 \in S$, then $\overline{\lambda}_S = 0$.

[24] Fix $S \in \mathbb{I}(x)$ and choose $y^S \in \varphi^S(x)$ together with any open set $V_S \subset X^S$ containing y^S. By lower semicontinuity, there exist an open neighborhood W_S of x with $\varphi^S(x') \cap V_S \neq \emptyset$ for every $x' \in W_S$. Thus $\varphi^S(x') \neq \emptyset$ on W_S. Taking $W := \bigcap_{S \in \mathbb{I}(x)} W_S$ yields $\mathbb{I}(x) \subseteq \mathbb{I}(x')$ for all $x' \in W$.

From (7.1):

$$\overline{x}_i = \sum_{S \in \mathbb{I}(i)} \overline{\lambda}_S \overline{z}_i^S.$$

Using Scarf's decomposition (Scarf 1971, pp. 178–179), we write:

$$\overline{x} = \sum_{S \in \mathbb{I}(i_0)} \overline{\lambda}_S y(S),$$

where $y(S) \in X$ satisfies:

$$y_i(S) = \begin{cases} \overline{z}_i^S, & \text{if } i \in S, \\ \Big(\sum_{E \in \mathbb{I}(i,-i_0)} \overline{\lambda}_E \overline{z}_i^E\Big) / \Big(\sum_{E \in \mathbb{I}(i,-i_0)} \overline{\lambda}_E\Big), & \text{if } i \notin S, \end{cases}$$

with $\mathbb{I}(i,-i_0) := \{E \in \mathbb{I} \,:\, i \in E \quad \text{and} \quad i_0 \notin E\}$.

By Claim 7.1:

$$\overline{x} = \sum_{S \in \mathbb{I}(i_0) \cap \mathbb{I}(\overline{x})} \overline{\lambda}_S y(S).$$

Each $y(S)$ satisfies $\pi^S(y(S)) = \overline{z}^S$, and from (7.2), we get $y(S) \in \operatorname{co} P_{i_0}(\overline{x})$. Thus $\overline{x} \in \operatorname{co} P_{i_0}(\overline{x})$, contradicting Assumption 2.3. □

7.2 The General Case

Proof of the Main Theorem. Fix, for each $S \in \mathbb{I}$, an anchor $\underline{x}^S \in F^S(X)$ and let $\mathcal{H}$ be the set of finite–dimensional subspaces $H \subset L$ with $\underline{x}^S \in H^S$ for all S. This ensures $\mathcal{H} \neq \emptyset$ and, for each $H \in \mathcal{H}$, the restricted feasible ranges $F_H^S(X)$ are nonempty.

For each $H \in \mathcal{H}$, define the restricted game

$$\mathcal{G}^H := \big(H, (X_i^H, P_i^H)_{i \in I}, F_H, (F_H^S)_{S \in \mathbb{I}}\big),$$

with $X_i^H := X_i \cap H$, $F_H^S(x) := F^S(x) \cap H^S$, $F_H(x) := F(x) \cap H^I$, and $P_i^H(x) := P_i(x) \cap H^I$. The restrictions preserve Assumptions 2.1, 2.2, 2.3, and 2.4, and balancedness is inherited by intersection with H. By Proposition 7.1, each $\mathcal{G}^H$ has an α–core element $x_H \in F_H$.

By compactness of $F \subset X$, the net $(x_H)_{H \in \mathcal{H}}$ has a subnet converging to some $\overline{x} \in F$.

Suppose $\overline{x}$ does not belong to the α–core. Then there exist $S \in \mathbb{I}$ and $y^S \in F^S(\overline{x}) \cap P^S(\overline{x})$.

Consider the cofinal directed subset

$$\mathcal{H}(y^S) := \{H \in \mathcal{H} \colon y^S \in H^S\}.$$

Passing to a subnet indexed by $\mathcal{H}(y^S)$ (and not relabeling), we still have $x_H \to \overline{x}$. By Proposition A.2, the correspondence $x \mapsto F^S(x) \cap P^S(x)$ is lower semicontinuous at $\overline{x}$. Hence, fixing any open neighborhood U_S of y^S in X^S, there exist an open neighborhood U of $\overline{x}$ in X such that

$$\big(F^S(x) \cap P^S(x)\big) \cap U_S \neq \emptyset, \quad \text{for all } x \in U.$$

For each $H \in \mathcal{H}(y^S)$ set $V_H := U_S \cap H^S$, which is open in the subspace topology of H^S and contains y^S. Then, for every $x \in U$,

$$\big(F^S_H(x) \cap P^S_H(x)\big) \cap V_H = \big(F^S(x) \cap H^S\big) \cap \big(P^S(x) \cap H^S\big) \cap V_H \neq \emptyset,$$

so $F^S_H(x) \cap P^S_H(x) \neq \emptyset$.

Since $x_H \to \overline{x}$ and U is a neighborhood of $\overline{x}$, we have $x_H \in U$ for all H large enough in $\mathcal{H}(y^S)$. For such H,

$$F^S_H(x_H) \cap P^S_H(x_H) \neq \emptyset,$$

which contradicts that x_H lies in the α–core of $\mathcal{G}^H$. Therefore $\overline{x}$ belongs to the α–core of $\mathcal{G}$.

□

In his concluding remarks, Kajii (1992) claimed that a Bewley-type limit argument cannot apply to the α_{K}-core due to lack of uniformity in blocking strategies. This does not contradict our result because our definition of $P^S(x)$ uses the compact set $W^{I \setminus S}$, unlike $P^S_{\mathrm{K}}(x)$, which uses $X^{I \setminus S}$ and may lack compactness. Hence, our intermediate α-blocking concept is crucial—it allows us to prove a stronger result under weaker topological assumptions.

8 Concluding Remarks

This paper establishes a general nonemptiness result for the α-core in games, encompassing and unifying existing results in the literature, including those of Scarf (1971), Florenzano (1989), and Kajii (1992). Our approach introduces an intermediate notion of blocking power that allows us to prove existence results under weaker topological assumptions than previously required. In particular, we relax the norm-compactness assumptions that limited earlier work and allow for nonordered preferences with externalities in infinite-dimensional strategy spaces.

We demonstrated the applicability of our main result in two novel economic settings: games with Knightian uncertainty and games with voting preferences. In both cases, the α-core is shown to be non-empty under conditions for which existing core existence results do not apply.

Overall, our contribution provides both theoretical insights and practical tools for advancing the analysis of cooperative behavior in complex economic environments, particularly those characterized by uncertainty, externalities, and strategic interdependence.

A Appendix

A.1 Continuity of Correspondences

We recall standard definitions and notations for correspondences between topological spaces. Let X and Y be topological spaces, and let P be a correspondence from X to Y. We say that P is:

(a) *lower semi-continuous* if, for every open set $V \subseteq Y$, the set $\{x \in X : P(x) \cap V \neq \emptyset\}$ is open;

(b) *upper semi-continuous* if, for every open set $V \subseteq Y$, the set $\{x \in X : P(x) \subseteq V\}$ is open;

(c) *continuous* if it is both lower and upper semi-continuous;

(d) said to have *open lower sections* if, for every $y \in Y$, the set $\{x \in X : y \in P(x)\}$ is open;

(e) said to have an *open (resp. closed) graph* if the set gph $P := \{(x, y) \in X \times Y : y \in P(x)\}$ is open (resp. closed).

We also state a fixed-point result due to Gourdel (1995), which plays a key role in the proof of our main result.

Lemma A.1 (Gourdel (1995))**.** *Let $X = \prod_{k=1}^{m+n} X_k$, where each X_k is a non-empty, compact, convex subset of a finite-dimensional Euclidean space. Suppose, for each k, that $\varphi_k : X \rightrightarrows X_k$ is a correspondence with convex (possibly empty) values. Assume that φ_k is lower semi-continuous for $k = 1, \ldots, m$ and upper semi-continuous with closed values for $k = m+1, \ldots, m+n$. Then there exists $\overline{x} \in X$ such that for all k, either $\varphi_k(\overline{x}) = \emptyset$ or $\overline{x}_k \in \varphi_k(\overline{x})$.*

A.2 Intermediate Results

We provide several technical results that serve as intermediate steps in the proof of the main result.

Proposition A.1. Under Assumptions 2.1, 2.2, and 2.4, the correspondence P^S from X to X^S has open lower sections for every coalition $S \in \mathbb{I}$.

Proof. Fix a coalition $S \in \mathbb{I}$ and recall

$$P^S(x) = \{y \in X^S \colon \{y\} \times W^{I \setminus S}(x) \subset \textstyle\bigcap_{i \in S} P_i(x)\}.$$

Case (i): each P_i has an open graph. To prove that P^S has open lower sections, we fix a social state $x \in X$ and $y \in P^S(x)$. Following standard arguments, the correspondence $x \mapsto \cap_{i \in S} P_i(x)$ also has an open graph. This implies that, for every $w \in\in W^{I \setminus S}(x)$, there exist open sets $U_w \ni x$, $V_w \ni y$ and $W_w \ni w$ such that $(y', w') \in \cap_{i \in S} P_i(x')$ for every $(x', y', w') \in U_w \times V_w \times W_w$. By compactness of $W^{I \setminus S}(x)$, there exists a finite set $F \subseteq W^{I \setminus S}(x)$ such that

$$W^{I \setminus S}(x) \subseteq W := \bigcup_{w \in F} W_w.$$

Let $U := \cap_{w \in F} U_w$ and $V := \cap_{w \in F} V_w$. The sets U and V are open and satisfy $U \ni x$ and $V \ni y$. Standard results in set-valued analysis guarantee that the correspondence $W^{I \setminus S}$ is continuous with compact values. Therefore, shrinking U if needed,

$$W^{I\setminus S}(x') \subseteq W, \quad \text{for all } x' \in U.$$

We deduce that

$$\{y'\} \times W^{I\setminus S}(x') \subseteq \bigcap_{i\in S} P_i(x')$$

for every $(x', y') \in U \times V$, proving that the graph of P^S is open and, in particular, that P^S has open lower sections.

Case (ii): weak externalities and open lower sections. Assume that $P_i(x) = \widetilde{P}_i(x) \times X^{-i}$ with $\widetilde{P}_i : X \to X_i$ having open lower sections, and each feasibility correspondence F^S is constant.[25] Observe that

$$P^S(x) = \prod_{i\in S} \widetilde{P}_i(x).$$

Products of correspondences with open lower sections have open lower sections, hence P^S has open lower sections.

Conclusion: in both cases P^S has open lower sections, as claimed. □

Proposition A.2. Under Assumptions 2.1, 2.2, and 2.4, the correspondence $x \mapsto F^S(x) \cap P^S(x)$ from X to X^S is lower semicontinuous for every coalition $S \in \mathbb{I}$.

Proof. Fix a coalition $S \in \mathbb{I}$, a social state $x \in X$, and an open set $W \subseteq X^S$ such that $F^S(x) \cap P^S(x) \cap W \neq \emptyset$.

Case (i): each P_i has an open graph. Pick $y \in F^S(x) \cap P^S(x) \cap W$. By Proposition A.1, P^S has an open graph. Hence there exist open sets $V \ni y$ and $U \ni x$ with

$$y' \in P^S(x'), \quad \text{for all } (x', y') \in U \times V.$$

Without any loss of generality, shrink V so that $V \subseteq W$. Since F^S is lower semicontinuous and $F^S(x) \cap V \neq \emptyset$, there exists an open set $U' \ni x$ such that

$$F^S(x') \cap V \neq \emptyset, \quad \text{for all } x' \in U'.$$

[25] For this proof, the constancy of F^S is not needed.

Shrinking U' is needed so that $U' \subseteq U$, we have $V \subseteq P^S(x') \cap W$ for all $x' \in U'$, hence

$$F^S(x') \cap P^S(x') \cap W \neq \emptyset, \quad \text{for all } x' \in U',$$

which prove lower semicontinuity at x.

Case (ii): weak externalities and open lower sections. Assume that $P_i(x) = \widetilde{P}_i(x) \times X^{-i}$ with $\widetilde{P}_i : X \to X_i$ having open lower sections, and each feasibility correspondence F^S is constant. Pick $y \in F^S \cap P^S(x) \cap W$, so $y_i \in \widetilde{P}_i(x)$ for every $i \in S$. For each $i \in S$, openness of the lower section at y_i yields an open $U_i \ni x$ with $y_i \in \widetilde{P}_i(x')$ for all $x' \in U_i$. Let $U := \bigcap_{i \in S} U_i$ (open, $x \in U$). Then $y \in P^S(x')$ for any $x' \in U$, and since F^S is constant and $y \in W$, we get

$$F^S \cap P^S(x') \cap W \neq \emptyset, \quad \text{for all } x' \in U,$$

establishing lower semicontinuity at x.

In both cases the correspondence $x \mapsto F^S(x) \cap P^S(x)$ is lower semicontinuous, as claimed. □

A.3 Proof of Claim 7.1

Recall that Σ is the simplex of $\mathbb{R}^{\mathbb{I}}$, defined by

$$\Sigma := \left\{ \mu = (\mu_S)_{S \in \mathbb{I}} \in \mathbb{R}^{\mathbb{I}} \;:\; \mu_S \geqslant 0 \;\forall S, \; \sum_{S \in \mathbb{I}} \mu_S = 1 \right\},$$

and Δ is the set of balanced weights:

$$\Delta := \left\{ \lambda = (\lambda_S)_{S \in \mathbb{I}} \in \mathbb{R}^{\mathbb{I}} \;:\; \lambda_S \geqslant 0 \;\forall S, \; \sum_{S \in \mathbb{I}(i)} \lambda_S = 1 \;\forall i \in I \right\}.$$

Let $\overline{\mu} \in \Sigma$ be such that $\overline{\mu}_S = 0$ for every $S \notin \mathbb{J}$, where $\mathbb{J} \subseteq \mathbb{I}$ is non-empty. Let $\overline{\lambda} \in \arg\max\{\overline{\mu} \cdot \lambda \;:\; \lambda \in \Delta\}$. By the Kuhn–Tucker Theorem, there exist multipliers $(\alpha_S)_{S \in \mathbb{I}} \in \mathbb{R}^{\mathbb{I}}_+$ and $(\eta_i)_{i \in I} \in \mathbb{R}^I$ such that

$$\forall S \in \mathbb{I}, \quad \overline{\mu}_S + \alpha_S + \sum_{i \in S} \eta_i = 0, \quad \text{and} \quad \alpha_S \cdot \overline{\lambda}_S = 0. \tag{A.1}$$

Since singletons are coalitions, taking $S = \{i\}$ in (A.1) gives $\eta_i \leqslant 0$ for all $i \in I$. Moreover, there exists $i_0 \in I$ such that $\eta_{i_0} < 0$. Indeed, since $\overline{\mu} \in \Sigma$, there exists $S_0 \in \mathbb{J}$ such that $\overline{\mu}_{S_0} > 0$. Using (A.1) for $S = S_0$, it follows that $\sum_{i \in S_0} \eta_i < 0$, so at least one η_i must be negative.

Now take any $S \notin \mathbb{J}$ such that $i_0 \in S$. Since $\overline{\mu}_S = 0$, (A.1) implies:

$$\alpha_S + \sum_{i \in S} \eta_i = 0.$$

Because each $\eta_i \leqslant 0$ and $\eta_{i_0} < 0$, the sum is strictly negative, which implies $\alpha_S > 0$, and thus $\overline{\lambda}_S = 0$ by complementary slackness.

Hence, for every $S \notin \mathbb{J}$ such that $i_0 \in S$, we have $\overline{\lambda}_S = 0$, completing the proof.

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